CHAPTER NINE

Sim, Chomp and Race track

New mathematical games of a competitive type, demanding more intellectual skill than luck, continue to proliferate both in the U.S. and abroad. In Britain they have become so popular that a monthly periodical called *Games and Puzzles* was started in 1972 just to keep devotees informed. *Strategy and Tactics* (a bimonthly with offices in New York City) is primarily concerned with games that simulate political or military conflicts, but a column by Sidney Sackson reports on new mathematical games of all kinds. Sackson's book *A Gamut of Games* (Random House, 1969) has a bibliography of more than 200 of the best mathematical board games now on the market.

Simulation games are games that model some aspect of human conflict: war, population growth, pollution, marriage, sex, the stock market, elections, racism, gangsterism—almost anything at all. They are being used as teaching devices, and some notion of how widely can be gained from the fact that a 1973 catalogue, *The Guide to Simulation Games for Education and Training,* by David W. Zuckerman and Robert E. Horn, runs to 500 pages.

We will take a look at three unusual new mathematical games. None requires a special board or equipment; all that is needed are pencil and paper (graph paper for the first game) and (for the third) a supply of counters.

Race Track, virtually unknown in this country, is a truly remarkable simulation of automobile racing. I do not know who invented it. It was called to my attention by Jurg Nievergelt, a computer scientist at the University of Illinois, who picked it up on a recent trip to Switzerland.

The game is played on graph paper. A racetrack wide enough to accommodate a car for each player is drawn on the sheet. The track may be of any length or shape, but to make the game interesting it should be strongly curved *[see Figure 64]*. Each contestant should have a pencil or pen of a different color. To line up the cars, each player draws a tiny box just below a grid point on the starting line. In the example illustrated the track will take three cars, but for

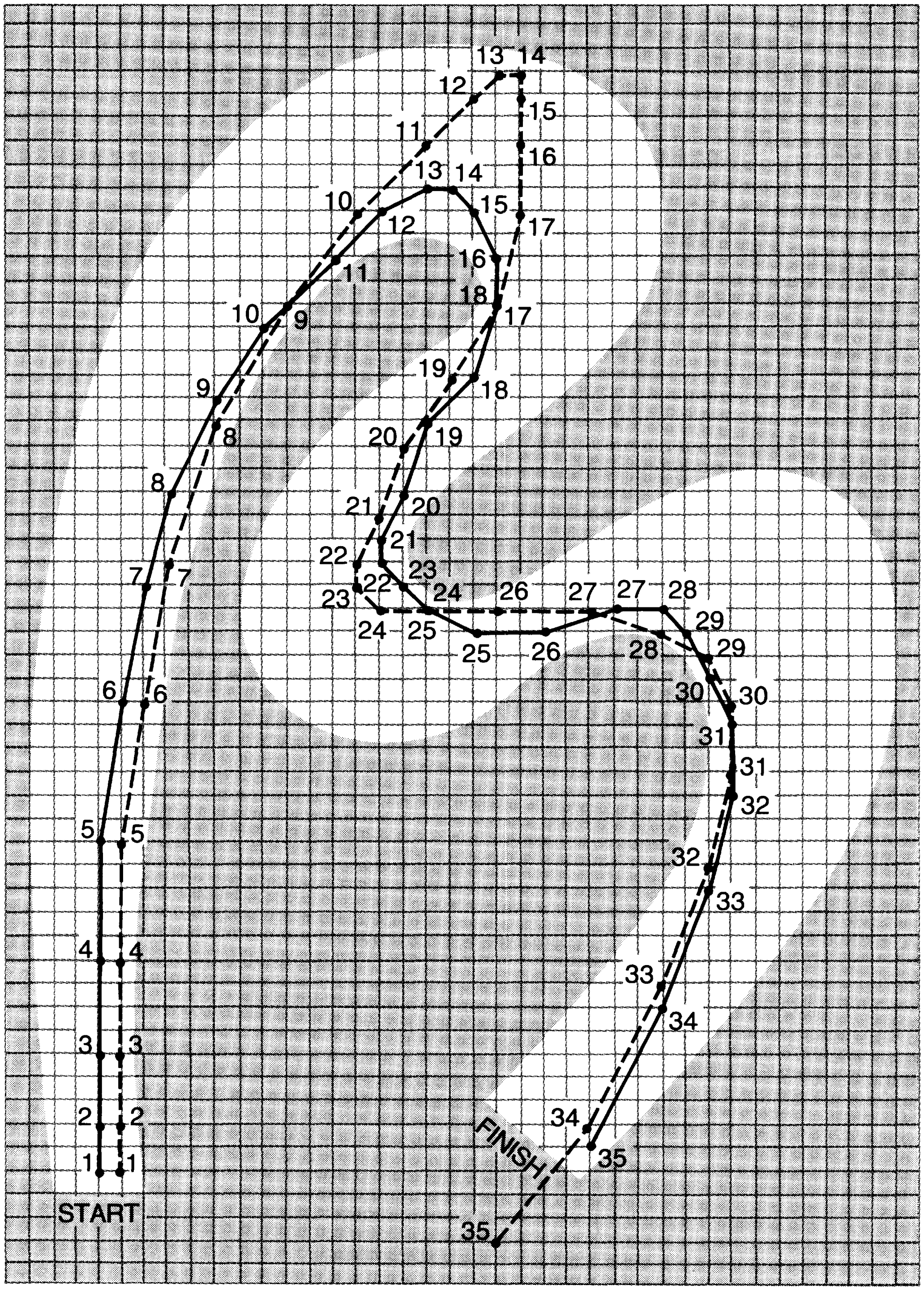

Figure 64 The Race Track game

simplicity a race of two cars is shown. Lots can be drawn to decide the order of moving. In the sample game, provided by Nievergelt, Black moves first.

You might suppose that a randomizing device now comes into play to determine how the cars move, but such is not the case. At each turn a player simply moves his car ahead along the track to a new grid point, subject to the following three rules:

1. The new grid point and the straight line segment joining it to the preceding grid point must lie entirely within the track.

2. No two cars may simultaneously occupy the same grid point. In other words, no collisions are allowed. For instance, consider move 22. Gray, the second player, would probably have preferred to go to the spot taken by Black on his 22nd move, but the no-collision rule prevented it.

3. Acceleration and deceleration are simulated in the following ingenious way. Assume that your previous move was k units vertically and m units horizontally and that your present move is k' vertically and m' horizontally. The absolute difference between k and k' must be either 0 or 1, and the absolute difference between m and m' must be either 0 or 1. In effect, a car can maintain its speed in either direction, or it can change its speed by only one unit distance per move. The first move, following this rule, is one unit horizontally or vertically, or both.

The first car to cross the finish line wins. A car that collides with another car or leaves the track is out of the race. In the sample game Gray slows too late to make the first turn efficiently. He narrowly avoids a crash, and the bad turn forces him to fall behind in the middle of the race. He takes the last curve superbly, however, and he wins by crossing the finish line one move ahead of Black. Neither driver, I should add, always makes his best moves.

Nievergelt programmed Race Track for the University of Illinois's Plato IV computer-assisted instruction system, which uses a new type of graphic display called a plasma panel. Two or three people can play against one another, or one person can play alone. The game became so popular that the authorities made it inaccessible for a week to prevent students from wasting too much time on it.

Our second pencil-and-paper game is called Sim, after Gustavus J. Simmons, a mathematician at the Sandia Corporation laboratories in Albuquerque, who invented it when he was working on his Ph.D. thesis on graph theory. He was not the first to think of it (the idea occurred independently to a number of mathematicians), but he was the first to publish it and to analyze it completely with a computer program. In his note titled "On the Game of Sim" (see the bibliography) he says that one of his colleagues picked the name as short for SIMple SIMmons, and because the game resembles the familiar game of nim.

Six points are placed on a sheet of paper to mark the vertexes of a regular hexagon. There are 15 ways to draw straight lines connecting a pair of points, producing what is called the complete graph for six points *[see Figure 65]*. Two

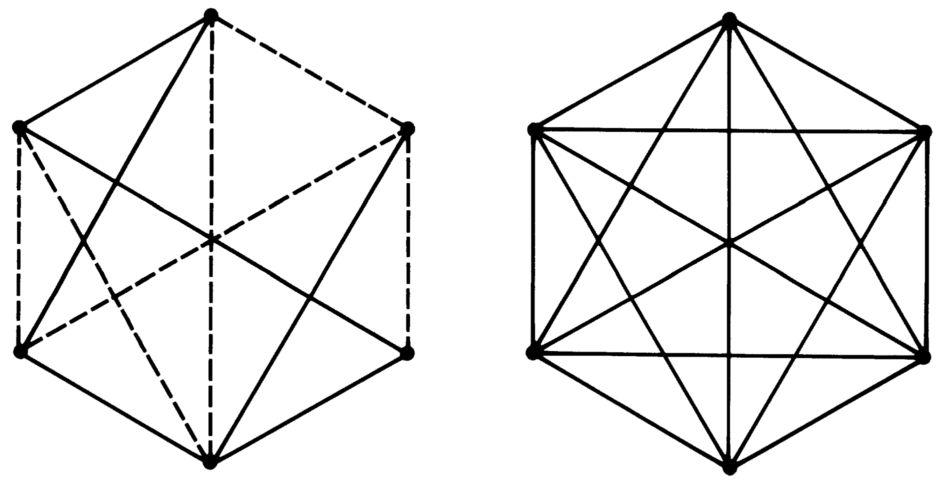

Figure 65 The game of Sim

Sim players take turns drawing one of the 15 edges of the graph, each using a different color. The first player to be forced to form a triangle of his own color (only triangles whose vertexes are among the six starting points count) is the loser.

If only two colors are used for the edges of a chromatic graph, it is not hard to prove that six is the smallest number of points whose complete chromatic graph is certain to contain a triangle with sides all the same color. Simmons gives the proof as follows: "Consider any vertex in a completely filled-in game. Since five lines originate there, at least three must be the same color — say blue. No one of the three lines joining the end points of these lines can be blue if the player is not to form a blue triangle, but then the three interconnecting lines form a red triangle. Hence at least one monochromatic (all one color) triangle must exist, and a drawn game is impossible."

With a bit more work a stronger theorem can be established. There must be at least *two* monochromatic triangles. A detailed proof of this is given by Frank Harary, a University of Michigan graph theorist, in his paper "The Two-Triangle Case of the Acquaintance Graph" *[see the bibliography]*. Harary calls it an acquaintance graph because it provides the solution to an old brainteaser: Of any six people, prove that at least three are mutual acquaintances or at least three are mutual strangers. Harary not only proves that there are at least two such sets but also shows that if there are exactly two, they are of opposite types (colors on the graph) if and only if the two sets have just one person (point) in common.

Because Sim cannot be a draw, it follows that either the first or the second player can always win if he plays correctly. When Simmons wrote his note in 1969, he did not know which player had the win, and in actual play among equally skillful players wins are about equally divided. Later he made an exhaustive computer analysis showing that the second player could always win. Because of symmetry, all first moves are alike. The computer results showed that the second player could respond by coloring any of the remaining 14 edges and still guarantee himself a win. (Actually, for symmetry reasons, there are only two fundamentally different second moves: one that connects with the first move, and one that does not.)

After the first player has made his second move, exactly half of the remaining plays lead to a sure win for the second player and half to a sure loss, assuming, of course, that both sides play rationally. If 14 moves are made without a win, the last move, by the first player, will always produce two monochromatic triangles of his color. This 14-move pattern is unique in the sense that all such patterns are topologically the same. Can you find a way of coloring 14 edges of the Sim graph, seven in one color and seven in another, so that there is no monochromatic triangle on the field?

The most interesting unanswered question about Sim is whether there is a relatively simple strategy by which the second player can win without having to memorize all the correct responses. Even if he has at hand a computer printout of the total game tree, it is of little practical use because it is enormously difficult to locate on the printout a position isomorphic to the one on the board. Simmons's computer results have been verified by programs written by Michael Beeler at the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology and, more recently by Jesse W. Croach, Jr., of West Grove, Pa., but no one has been able to extract from the game tree a useful mnemonic for the second player.

Sim can, of course, be played on other graphs. On complete graphs for three and four points the game is trivial, and for more than six points it becomes too complicated. The pentagonal five-point graph, however, is playable. Although a draw is possible, I am not aware of any proof that a draw is inevitable if both sides make their best moves.

Our third game, which I call Chomp, is a nim-type game invented by David Gale, a mathematician and economist at the University of California at Berkeley. Gale is the inventor of Bridg-it, a popular topological board game still on the market. What follows is based entirely on results provided by Gale.

Chomp can be played with a supply of counters *[see Figure 66]* or with O's or X's on a sheet of paper. The counters are arranged in a rectangular formation. Two players take turns removing counters as follows. Any counter is selected. Imagine that this counter is inside the vertex of a right angle through the field, the base of the angle extending east below the counter's row and its other side extending vertically north along the left side of the counter's column. All counters inside the right angle are removed. This constitutes a move. It is as though the field were a cracker and a right-angled bite were taken from it by jaws approaching the cracker from the northeast.

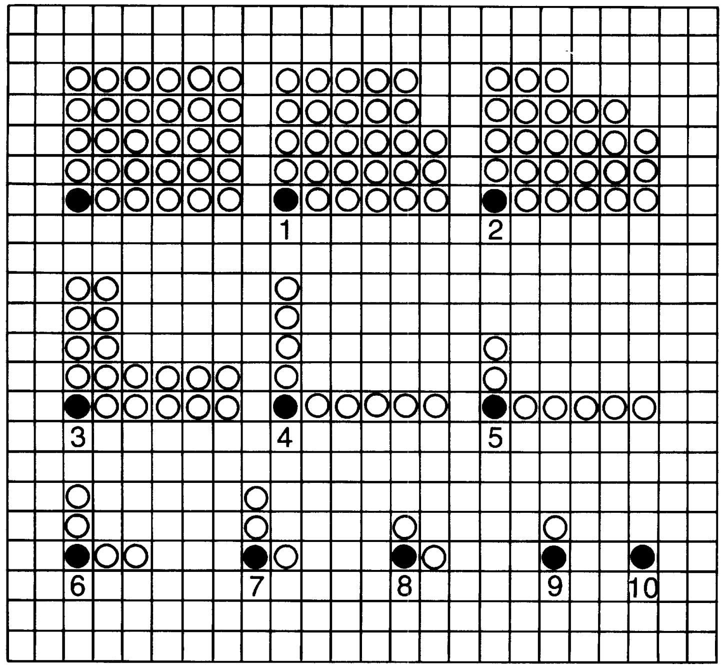

Figure 66 Chomp on a 5-by-6 field

The object of the game is to force your opponent to chomp the poison counter at the lower left corner of the array *[black counter]*. The reverse form of Chomp — winning by taking this counter — is trivial because the first player can always win on his first move by swallowing the entire rectangle.

What is known about this game? First, we dispose of two special cases for which winning strategies have been found.

1. When the field is square, the first player wins by taking a square bite whose side is one less than that of the original square. This leaves one column and one row, with the poison piece at the vertex *[see Figure 67, left]*. From now on the first player "symmetrizes." Whatever his opponent takes from either line, he takes equally from the other. Eventually the second player must take the poison piece.

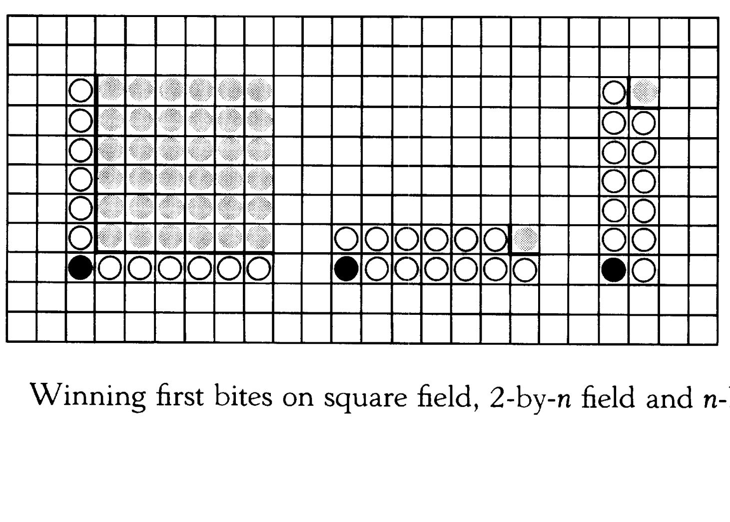

Figure 67 Winning first bites on square field, 2-by-*n* field and *n*-by-2 field

2. When the field is 2 by n, the first player can always win by taking the counter at top right *[see Figure 67, middle and right]*. Removing that counter leaves a pattern in which the bottom row has one more counter than the top row. From now on the first player always plays to restore this situation. One can easily see that it can always be done and that it ensures a win. The same strategy applies to fields of width 2, except now the first player always makes sure that the left column has one more counter than the right column.

With the exception of these two trivial cases, no general strategy for Chomp is known. There is, however, and this is what makes Chomp so interesting, a simple proof that the first player can always win. Like similar proofs that apply to Bridg-it, Hex, generalized ticktacktoe and many other games, the proof is nonconstructive in that it is of no use in finding a winning line of play. It only tells you that such a line exists. The proof hinges on taking the single counter at the upper right corner in the opening move. There are two possibilities: (1) It is a winning first move; (2) it is a losing first move. If it is a losing one, the second player can respond with a winning move. Put another way, he can take a bite that leaves a position that is a sure loss for the first player. But no matter how the second player bites, it leaves a position that the first player could have left if his first bite had been bigger. Therefore if the second player has a winning response to the opening move of taking the counter at top right, the first player could have won by a different opening move that left exactly the same pattern.

In short, either the first player can always win by taking the counter at top right, or he can always win by some other first move.

"We normally think of nonconstructive proofs in mathematics as being proofs by contradiction," Gale writes. "Note that the above proof is not of that type. We did not start by assuming that the game was a loss for the first player and then obtain a contradiction. We showed directly that there was a winning strategy for the first player. The word 'not' was never used in the argument. Of course we used implicitly the fact that any game of this kind is a win for either the first or the second player, but even the proof of this fact can be given by a simple inductive argument that does not use any law of the excluded middle."

This is essentially all that is known about Chomp except for some curious empirical results Gale obtained from a complete computer analysis of the 3-by-n game for all n's equal to or less than 100. In every case it turned out that the winning first move is unique. Figure 68 shows the winning moves for 3-high fields of widths 2 through 12. Rotating and reflecting these patterns give winning moves on 3-wide fields of heights 2 through 12 because any m-by-n game is symmetrically the same as the n-by-m game.

A winning first move on a 3-high field must be one or two rows deep. (A 3-deep bite would leave a smaller rectangle and thus throw the win to the

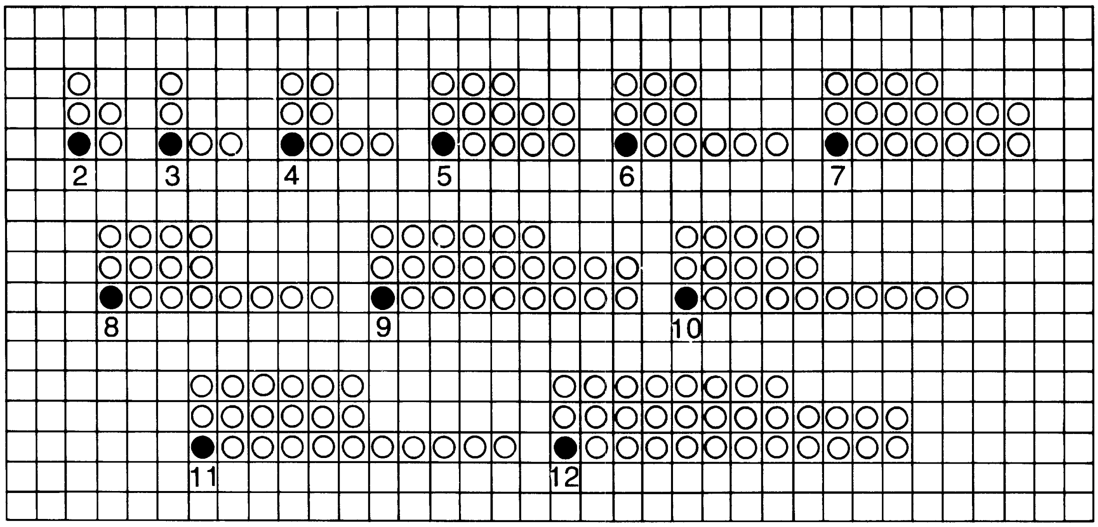

Figure 68 Winning first bites on 3-by-*n* fields

second player.) Roughly 58 percent of the winning first moves are two rows deep, and 42 percent are one row deep. Note that the 1-row moves either stay the same or increase in width as *n* increases, and the same is true of the 2-row moves. A partial analysis of all 3-high fields with widths less than 171 showed that the sole exception to this rule occurs when *n* is 88. The winning first move on the 3-by-88 rectangle is 2 by 36, which is one unit less wide than the winning 2-by-37 move on the 3-by-87 field. "Phenomena like this," Gale writes, "lead one to believe that a simple formula for the winning strategy might be quite hard to come by."

There are two outstanding unproved conjectures:

1. There is only one winning first move on all fields.

2. Taking the counter at the top right corner always loses except on 2-by-*n* (or *n*-by-2) fields.

The second conjecture has been established only for fields with widths or heights of 3. Readers are invited to discover the unique winning openings on 4-by-5 and 4-by-6 rectangles.

ANSWERS

Sim has only one basic position (variants are topologically identical) that allows the game to go 14 moves without a monochrome triangle *[see Figure 69, top].* The 4-by-5 and 4-by-6 fields for Chomp are won by the unique first moves shown at the bottom of Figure 69.

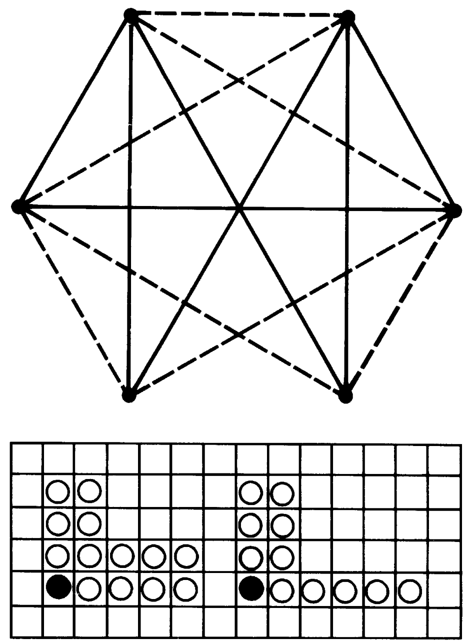

Figure 69 Sim game that ends on move 15 and winning chomps

David Gale, who invented Chomp, has considered the game on infinite rectangular arrays. Readers may enjoy proving (on the basis of the given theorems) that the first player wins on n-by-infinity fields (provided that n is not 2) and on infinity-by-infinity squares but loses on 2-by-infinity arrays.

ADDENDUM

The three games prompted a variety of interesting letters. Many readers felt that Race Track rules should not allow one car to win if another car on the same move could also cross the finish line. They suggested giving the win to the car farthest from the finish line at the end of the move. Joe Crowther was the first of many readers who proposed drawing one or two patches on the roadway to represent oil slicks. Cars are required to move at a constant speed and direction when passing wholly or partly through each patch. J. P. Schell, in addition to oil slicks, proposed adding upgrades and downgrades to force cars to speed up or slow down, as well as stationing pretty girls along the track to distract drivers. Others suggested adding pit stops here and there and requiring a driver to lose one move by coming to zero velocity within any one pit of his choice. Some readers thought it would simplify the game if the finish line were always drawn along one of the grid lines.

David Pope suggested a fast-acceleration move. Whenever a car slows to a full stop, it can, on the next move, go any desired distance in either or both of the two directions. Tom Gordon, who welcomed the game as a teaching device for his high school physics students, added a power-braking option that allows a car to reduce both coordinates by two units, provided the move continues the preceding move in a straight line.

C. R. S. Singleton described two novel variants of the game: (1) Instead of a track, numbered gates are marked on the graph. Cars must pass through the gates in numerical order. (2) A series of numbered checkpoints are substituted for the track. Cars must visit each checkpoint by ending a move on that point.

Michael D. Greenberg and his friends at the Westinghouse Aerospace Division in Baltimore adopted two rules to offset the advantage of a first move: (1) Slant the starting line (as in actual racing) and allow the second player to choose between the two starting points. (2) Allow cars to occupy the same point at the same time. They also preferred to draw the track along grid lines to avoid arguments over whether a point was on the track or inside it. Two British readers, Giles Vaughan-Williams and John Kinory, devised rules allowing cars to brake and skid when rounding a sharp curve at high speed.

I have been unable to determine the origin of Race Track. A car-race game very similar to it appeared under the name of *Le Zip* in a French book by Pierre Berloquin, *Le Livre des Jeux,* published about 1971. It is reprinted in Berloquin's book *100 Jeux de Table* (Paris, 1976). A version of *Race Track* appeared as game 13 in the Hewlett-Packard *Games Pac 1* book (1976) for use with the company's HP-67 and HP-97 calculators.

The game of Sim on a complete graph for five points is now known to be a draw if played rationally. (All draws are topologically equivalent to a pentagram of one color inscribed in a pentagon of the other color. Think of the points as balls connected by elastic strings. If one pattern can be changed to another, they are considered identical.) A complete game tree for five-point Sim was hand-constructed by Eugene A. Herman of Grinnell College and Leslie E. Shader of the University of Wyoming. Jesse W. Croach, Jr., of West Grove, Pa., was able to draw the tree by extracting information from his computer printout for six-point Sim. The first computer program written specifically for five-point Sim was by Ashok K. Chandra of the Artificial Intelligence Laboratory at Stanford University. It produced a complete tree in a few seconds. The results were confirmed by Michael Beeler's program.

Both Chandra and Herman noticed that a good strategy in five-point Sim is to form a closed circuit of four edges of your color, with a fifth edge attached to any of the four dots. This guarantees your win. Herman noticed that as soon as a dot has three edges of the same color attached to it a draw becomes impossible. Variations and generalizations of Sim came from several readers.

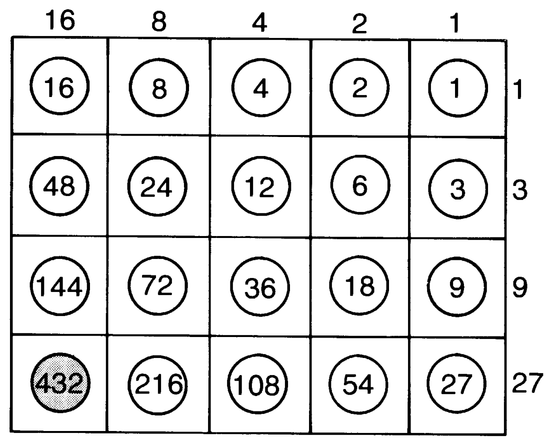

Figure 70 Chomp as a divisor game

The most surprising letter (to put it mildly) was from G. J. Westerink, of Veenendaal in the Netherlands, disclosing that the game of Chomp is isomorphic with a number game invented by the late Fred Schuh, a mathematician at Delft Technical College. It is one of the prettiest isomorphisms I have ever encountered in recreational mathematics. The game does not appear in Schuh's *Master Book of Mathematical Puzzles and Recreations* (Dover, 1968), but he explained it in a 1951 paper cited in the bibliography. Two players agree on any positive integer, N. A list is made of all the divisors (including N and 1); then players take turns crossing out a divisor and all *its* divisors. The person forced to take N loses. Planar Chomp corresponds to this game when N has exactly two prime divisors, solid Chomp to the game when N has three prime divisors, four-dimensional Chomp when N has four prime divisors and so on.

This is best made clear with an example. Consider $N = 432$, a number that prime-factors to $2^4 \times 3^3$. Draw a rectangular Chomp field with sides of 5 and 4 (the exponents raised by 1), and label the four rows with powers of 3 and the five columns with powers of 2. Counters have values that are products of their row and column *[see Figure 70]*. The equivalence of Chomp to the divisor game is now readily apparent. Moreover, any integer whose prime factors have the formula $m^4 \times n^3$ will correspond to the same Chomp field. Incredibly, most of the theorems discovered by David Gale for his game of Chomp (including the beautiful proof of first-player win) had been discovered by Schuh in arithmetical form!

Schuh offered to play readers by correspondence, using $N = 720$. Because the factors of 720 are $2^4 \times 3^2 \times 5^1$, it corresponds to Chomp on a 5-by-3-by-2 field. This proved to have two winning first moves (counters 36 and 48 when numbered according to the system explained). Like Gale, Schuh was unable to find a strategy for first-player win, a way to determine a winning first move short of constructing the game tree or a two-prime (planar Chomp) game that had more than one winning first move.

The first counterexample to the conjecture that all planar games have unique winning first moves was found by Ken Thompson of Bell Laboratories. His computer program produced many examples of fields with two first-move wins, the smallest being 8 by 10. The winning moves leave either five columns of 8 and five of 4, or eight columns of 8 and two of 3. This has been confirmed by Beeler. In 1981 Gil Golani, a student of Z. Wakeman, a mathematician at Ben-Gurion University of the Negev, Israel, wrote a PASCAL program that found a violation of the conjecture on an even smaller board. The two winning first moves for a 6-by-13 rectangle are to take two columns of three lines or five columns of two lines.

Cubical Chomp is an interesting challenge. It is easily seen that the winning first move on the order-2 cube is to take the order-1 cube from the corner. Westerink's analysis of the order-3 cube reveals that the winning first move is to take an order-2 cube from the corner. I previously gave Gale's simple proof that the winning move for any square of order n is to take a square of order $n - 1$. Do winning first moves on all cubes of order n consist in taking a cube of order $n - 1$? If so, does this generalize to n-space cubes?

David Gale reported that in three-dimensional chomp a 2-by-m-by-n game is a trivial win for the first player even when m or n or both are infinite. The first player simply leaves a 2-by-infinity field—a loss for the opponent. The 3 by 3 by 3 and the 3 by 3 by infinity apparently are still unsolved.

In a later letter Gale reported the following result. Suppose the initial field has any finite number of counters in each row but an infinite number in the bottom row. Regardless of the pattern, the game is a win for the first player. Moreover, the winning first move is unique. Gale sent an ingenious nonconstructive proof by contradiction. As in his previous proof of first-player win in standard Chomp, it does not provide what the winning first move is.

Alan Barnert, a Manhattan ophthalmologist and friend, made a right-angled scoop for playing Chomp with a field of raisins. Players ate the raisins scooped on each move until only the poison raisin remained.

David Klarner, in "How to Be a Winner" (see the bibliography), explains how to draw a directed "state graph" that makes visually clear exactly how a first player wins a game of Chomp. Figure 71 shows such a graph for the 2-by-3 game. On the left the actual positions after each move are shown. Arrows indicate possible transitions. On the right is a more abstract diagram of the same graph. Winning states are black. The first player's strategy is to make the initial black move at the top, then to always play to a winning (black) state on his next move. He is certain to reach the final winning state at the bottom. For larger fields, of course, such graphs quickly become too complex to draw.

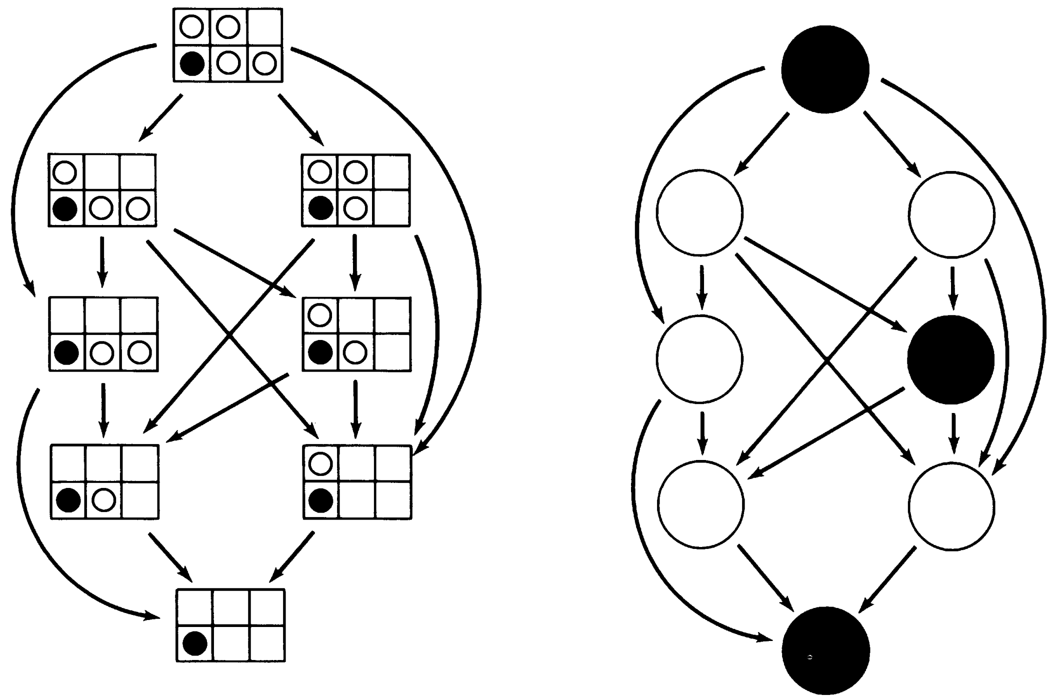

Figure 71 A state graph for 2 × 3 chomp

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