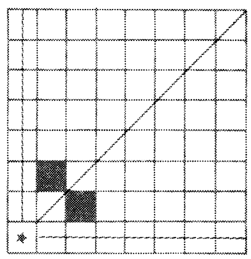
CHAPTER 8

Wythoff's Nim



An analysis of a simple two-person game can lead into fascinating corners of number theory. We begin with a charming, little-known game played on a chessboard with a single queen. Before we are through, we shall have examined a remarkable pair of number sequences that are intimately connected with the golden ratio and generalized Fibonacci sequences.

The game, which has no traditional name, was invented about 1960 by Rufus P. Isaacs, a mathematician at Johns Hopkins University. It is described briefly (without reference to chess) in Chapter 6 of the 1962 English translation of *The Theory of Graphs and Its Applications*, a book in French by Claude Berge. (We met Berge in the previous chapter as a member of the Oulipo.) Let's call the game "Corner the Lady."

Player *A* puts the queen on any cell in the top row or in the column farthest to the right of the board; the cells appear in gray in Figure 48. The queen moves in the usual way but only west, south or southwest. Player *B* moves first, then the players alternate moves. The player who gets the queen to the starred cell at the lower left corner is the winner.

No draw is possible, so that *A* or *B* is sure to win if both sides play rationally. It is easy to program an HP-97 printing calculator or the HP-67 pocket calculator to play a perfect game. Indeed, a magnetic card

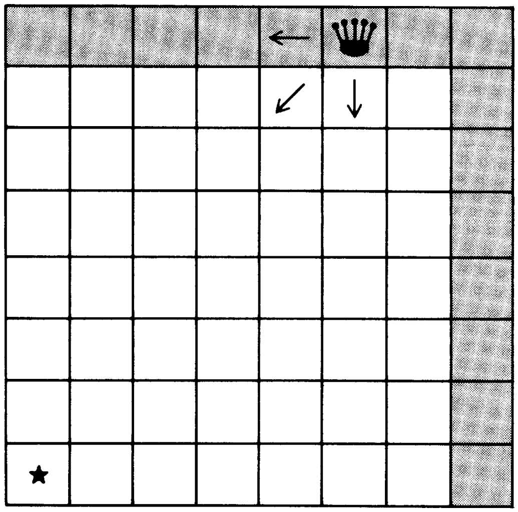


Figure 48
The cornering game of Rufus P. Isaacs

supplied with Hewlett-Packard's book *HP-67/HP-97 Games Pac 1* provides just such a program.

Isaacs constructed a winning strategy for cornering the queen on boards of unbounded size by starting at the starred cell and working backward. If the queen is in the row, column or diagonal containing the star, the person who has the move can win at once. Mark these cells with three straight lines as is shown in part A of Figure 49. It is clear that the two shaded cells are "safe," in the sense that if you occupy either one, your opponent is forced to move to a cell that enables you to win on the next move.

Part B of the illustration shows the next step of our recursive analysis. Add six more lines to mark all the rows, columns and diagonals containing the two previously discovered safe cells. This procedure allows us to shade two more safe cells as shown. If you occupy either one, your opponent is forced to move, so that on your next move you can either win at once or move to the pair of safe cells nearer the star.

Repeating this procedure, as is shown in part C of the illustration, completes the analysis of the chessboard by finding a third pair of safe cells. It is now clear that Player *A* can always win by placing the queen on the shaded cell in either the top row or the column farthest to the right. His strategy thereafter is simply to move to a safe cell, which he can always do. If *A* fails to place the queen on a safe cell, *B* can always win by the same strategy. Note that winning moves are not necessarily unique. There are times when the player with the win has two choices; one may delay the win, the other may hasten it.

Our recursive analysis extends to rectangular matrixes of any size or shape. In part D of the illustration, a square with 25 squares on a side is

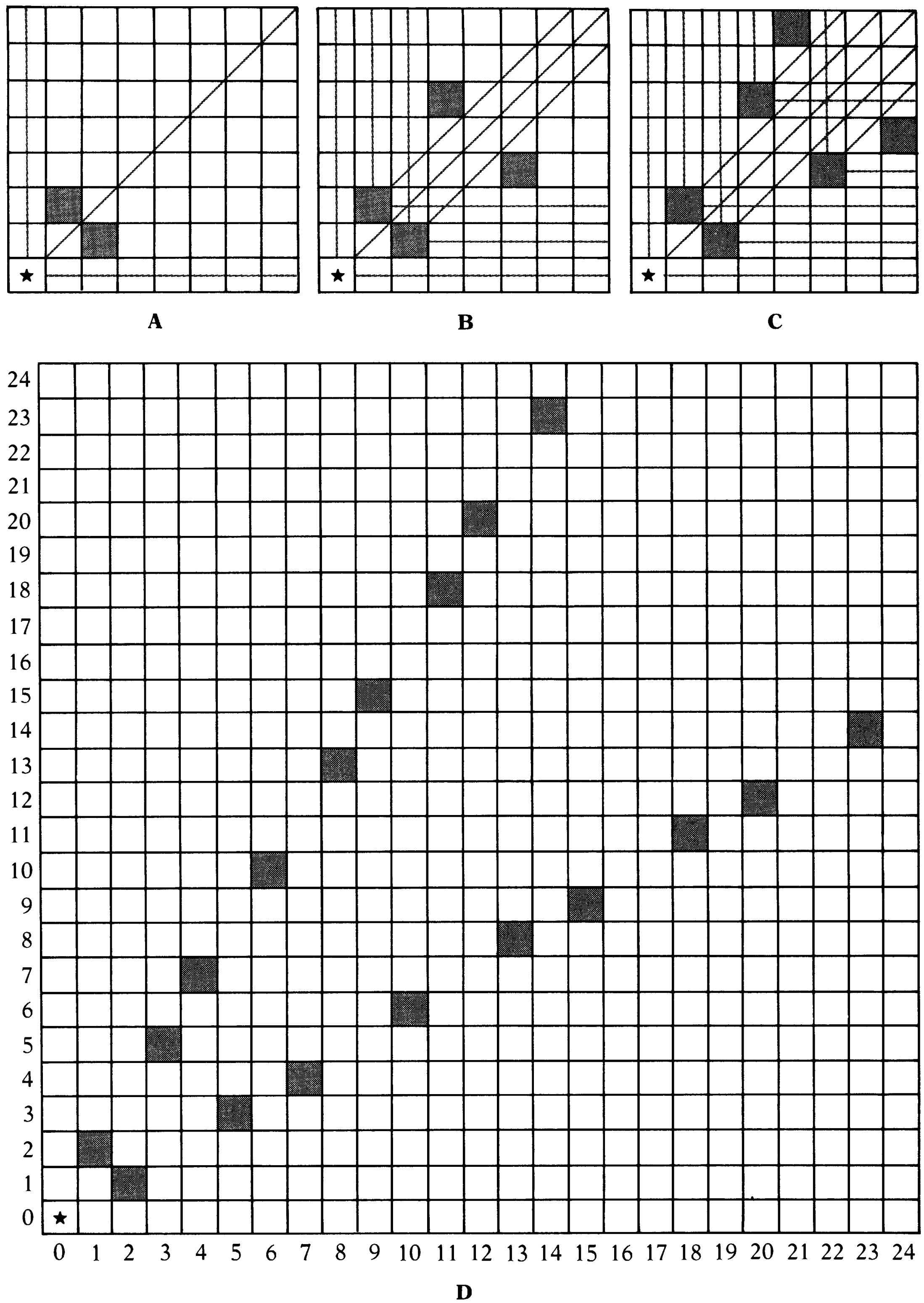


Figure 49 *(A, B, C) A recursive analysis of "Corner the Lady" (D) The first nine pairs of safe cells*

shown with all the safe cells shaded. Note that they are paired symmetrically with respect to the main diagonal and lie almost on two lines that fan outward to infinity. Their locations along those lines seem to be curiously irregular. Are there formulas by which we can calculate their positions nonrecursively?

Before answering let us turn to an old counter take-away game said to have been played in China under the name *tsyan-shidzi*, which means "choosing stones." The game was reinvented by the Dutch mathematician W. A. Wythoff, who published an analysis of it in 1907. In Western mathematics it is known as "Wythoff's Nim."

The game is played with two piles of counters, each pile containing an arbitrary number of counters. As in Nim, a move consists in taking any number of counters from either pile. At least one counter must be taken. If a player wishes, he may remove an entire pile. A player may take from both piles (which he may not in Nim), provided that he takes the same number of counters from each pile. The player who takes the last counter wins. If both piles have the same number of counters, the next player wins at once by taking both piles. For that reason the game is trivial if it starts with equal piles.

We are ready for our first surprise. Wythoff's Nim is isomorphic with the Queen-Cornering game! When Isaacs invented the game, he did not know about Wythoff's Nim, and he was amazed to learn later that his game had been solved as early as 1907. The isomorphism is easy to see. As is shown in part D of Figure 49, we number the 25 columns along the x coordinate axis, starting with 0; the rows along the y coordinate axis are numbered the same way. Each cell can now be given an x/y number. These numbers correspond to the number of counters in piles x and y. When the queen moves west, pile x is diminished. When the queen moves south, pile y is diminished. When it moves diagonally southwest, both piles are diminished by the same amount. Moving the queen to cell 0/0 is equivalent to reducing both piles to 0.

The strategy of winning Wythoff's Nim is to reduce the piles to a number pair that corresponds to the number pair of a safe cell in the Queen game. If the starting pile numbers are safe, the first player loses. He is certain to leave an unsafe pair of piles, which his opponent can always reduce to a safe pair on his next move. If the game begins with unsafe numbers, the first player can always win by reducing the piles to a safe pair and continuing to play to safe pairs.

The order of the two numbers in a safe pair is not important. This condition corresponds to the symmetry of any two cells on the chessboard with respect to the main diagonal: they have the same coordinate numbers, one pair being the reverse order of the other. Let us take the safe pairs in sequence, starting with the pair nearest 0/0, and arrange them in a row with each smaller number above its partner, as in Figure 50. Above the pairs write their "position numbers." The top numbers of the safe pairs form a sequence we shall call *A*. The bottom numbers form a sequence we shall call *B*.

These two sequences, each one strictly increasing, have so many remarkable properties that dozens of technical papers have been written about them. Note that each *B* number is the sum of its *A* number and its position number. If we add an *A* number to its *B* number, the sum is an *A* number that appears in the *A* sequence at a position number equal to *B*. (An example is $8 + 13 = 21$. The 13th number of the *A* sequence is 21.)

We have seen how the two sequences are obtained geometrically by drawing lines on the chessboard and shading cells according to a recursive algorithm. Can we generate the sequences by a recursive algorithm that is purely numerical?

We can. Start with 1 as the top number of the first safe pair. Add this to its position number to obtain 2 as the bottom number. The top number of the next pair is the smallest positive integer not previously used. It is 3. Below it goes 5, the sum of 3 and its position number. For the top of the third pair write again the smallest positive integer not yet used. It is 4. Below it goes 7, the sum of 4 and 3. Continuing in this way will generate series *A* and *B*.

There is a bonus. We have discovered one of the most unusual properties of the safe pairs. It is obvious from our procedure that every positive integer must appear once and only once somewhere in the two sequences.

Is there a way to generate the two sequences nonrecursively? Yes. Wythoff was the first to discover that the numbers in sequence *A* are simply multiples of the golden ratio rounded down to integers! (He wrote that he pulled this discovery "out of a hat.")

The golden ratio, as most readers of this book are aware, is one of the most famous of all irrational numbers. Like pi it has a way of appearing in unlikely places. Ancient Greek mathematicians called it the "extreme and mean ratio" for the following reason. Divide a line segment into

Position (n)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A. $[n\phi]$	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24
B. $[n\phi^2]$	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39

Figure 50 *The first 15 safe pairs in Wythoff's Nim*

parts A and B in such a way that the ratio of length A to length B is the same as the ratio of the entire line to A. You have divided the line into a golden ratio. Because this has been widely thought to be the most pleasing way to divide a line, the golden ratio has provoked a bulky literature (much of it crankish) about the use of the ratio in art and architecture.

We can calculate the golden ratio by assigning a length of 1 to line segment B. Our method of dividing the line is expressed by $(A + 1)/A = A/1$, a simple quadratic equation that produces for A a positive value of $(1 + \sqrt{5})/2 = 1.61803398 \ldots$, the golden ratio. Its reciprocal is $0.61803398. \ldots$ It is the only positive number that becomes its own reciprocal when 1 is taken from it and that becomes its own square when 1 is added to it. Its negative reciprocal has the same properties. In Britain the golden ratio is usually signified by the Greek letter τ (tau). I shall follow the American practice of calling it ϕ (phi).

The numbers in sequence A are given by the formula $[n\phi]$, where n is the position number and the brackets signify discarding the fractional part. B numbers can be obtained by adding A numbers to their position numbers, but it turns out that they are rounded-down multiples of the square of phi. The formula for sequence B, therefore, is $[n\phi^2]$. The fact that every positive integer appears once and only once among the safe pairs can be expressed by the following remarkable theorem: The set of integers that lie between successive multiples of phi and between successive multiples of phi squared is precisely the set of natural numbers.

Two sequences of increasing positive integers that together contain every positive integer just once are called "complementary." Phi is not the only irrational number that generates such sequences, although it is the only one that gives the safe pairs of Wythoff's Nim. In 1926 Sam Beatty, a Canadian mathematician, published his astounding discovery that any positive irrational number generates complementary sequences.

Let k be the irrational number, with k greater than 1. Sequence A consists of multiples of k, rounded down, or $[nk]$, where n is the position number and the brackets indicate discarding the fraction. Sequence B consists of rounded-down multiples of $k/(k - 1)$, or $[nk/(k - 1)]$. Complementary sequences produced in this way are called Beatty sequences. If k is phi, the second formula gives rounded-down multiples of $1.618+/0.618+ = 2.618+$, which, owing to the whimsical nature of phi, is the square of phi. Readers might like to convince themselves that Beatty's formulas do indeed produce complementary sequences by letting $k = \sqrt{2}$, pi, e or any other irrational, and that rational values for k fail to produce such sequences.

Whenever the golden ratio appears, it is a good bet that Fibonacci numbers lurk nearby. The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34 . . . , in which each number after the first two is the sum of the two preceding numbers. A general Fibonacci sequence is defined in the same way, except that it can begin with any pair of numbers. A property of every Fibonacci sequence of positive integers is that the ratio of adjacent terms gets closer and closer to phi, approaching the golden ratio as a limit.

If we partition the primary Fibonacci sequence into pairs, 1/2, 3/5, 8/13, 21/34 . . . , it can be shown that every Fibonacci pair is a safe pair in Wythoff's Nim. The first such pair not in this sequence is 4/7. If we start another Fibonacci sequence with 4/7, however, and partition it 4/7, 11/18, 29/47 . . . , all these pairs are also safe in Wythoff's Nim. Indeed, these pairs belong to a Fibonacci sequence of what are called Lucas numbers that begins 2, 1, 3, 4, 7, 11. . . .

Imagine that we go through the infinite sequence of safe pairs (in the manner of Eratosthenes' sieve for sifting out primes) and cross out the infinite set of all safe pairs that are pairs in the Fibonacci sequence. The smallest pair that is not crossed out is 4/7. We can now cross out a second infinite set of safe pairs, starting with 4/7, that are pairs in the Lucas sequence. An infinite number of safe pairs, of which the lowest is now 6/10, remain. This pair too begins another infinite Fibonacci sequence, all of whose pairs are safe. The process continues forever. Robert Silber, a mathematician at North Carolina State University, calls a safe pair "primitive" if it is the first safe pair that generates a Fibonacci sequence. He proves that there are an infinite number of primitive safe pairs. Since every positive integer appears exactly once among the safe pairs, Silber concludes that there is an infinite sequence of Fibonacci sequences that exactly covers the set of natural numbers.

Take the primitive pairs 1/2, 4/7, 6/10, 9/15 . . . in order and write down their position numbers, 1, 3, 4, 6. . . . Does this sequence look familiar? As Silber shows, it is none other than sequence A. In other words, a safe pair is primitive if and only if its position number is a number in sequence A.

Suppose you are playing Wythoff's game with a very large number of counters or on a chessboard of enormous size. What is the best way to determine whether a position is safe or unsafe, and how do you play perfectly if you have the win?

You can, of course, use the phi formulas to write out a sufficiently large chart of safe pairs, but this is hard to do without a calculator. Is there a simpler way comparable to the technique of playing perfect Nim by writing the pile numbers in binary notation? Yes, there is, but it uses a more eccentric type of number representation called Fibonacci notation that has been intensively studied by Silber and his colleague Ralph Gellar and also by other mathematicians such as Leonard Carlitz of Duke University.

Write the Fibonacci sequence from right to left as is shown in Figure 51. Above it number the positions from right to left. With the aid of this chart we can express any positive integer in a unique way as the sum of Fibonacci numbers. Suppose we want to write 17 in Fibonacci notation. Find the largest Fibonacci number that is not greater than 17 (it is 13) and put a 1 below it. When we move to the right, we find the next number that, added to 13, gives a sum that does not exceed 17. It is 3, and so a 1 goes below 3. When we move to the right again, the next number that gets a 1 is the 1 in the second position. The unused Fibonacci numbers get 0's.

The result is 1001010, a unique representation of 17. To translate it back to decimal notation sum the Fibonacci numbers indicated by the positions of the 1's: $13 + 3 + 1 = 17$. The 1 farthest to the right in the Fibonacci sequence is never used, so that all numbers in Fibonacci notation end in 0. It is also easy to see there are never two adjacent 1's. If there were, they would have a sum equal to the next Fibonacci number on the left, and our rules would give that number a 1 and give 0's to the original pair of adjacent 1's.

In Fibonacci notation the sum of a safe pair is the *B* number with 0 appended. From this it follows that the Fibonacci sequence is obtained by starting with 10 and adding 0's: 10, 100, 1000, 10000. . . . The same procedure gives any Fibonacci sequence generated by a primitive pair. For example, the Lucas sequence starting with 4/7 is 1010, 10100, 101000, 1010000. . . .

Every *A* number in Fibonacci notation has the 1 farthest to the right at an even position from the right. Every *B* number is obtained by adding 0 to the right of its *A* partner. Therefore every *B* number has the 1 farthest to the right in an odd position. Since every counting number is either an *A* number or a *B* number, we have a simple way of deciding

. . . 10	9	8	7	6	5	4	3	2	1		
. . . 55	34	21	13	8	5	3	2	1	1		
			1	0	0	1	0	1	0	=	17

Figure 51 *Fibonacci notation for 17*

whether a given position in Wythoff's Nim is safe or unsafe. Write the two numbers in Fibonacci notation. If the smaller one is an A number, and if adding 0 produces the other number, the position is safe; otherwise it is unsafe.

An example of the method is $8/13 = 100000/1000000$. The 1 in 100000 is at position 6, an even position, so that 100000 is an A number. Adding 0 produces $1000000 = 13$, the partner of 8. We know that 8/13 is safe. If it is your turn, your opponent has the win. If you think he cannot play perfectly, make a small random move and hope that he soon will make a mistake.

If the pair is unsafe and it is your turn, how can you determine the safe position to which you must play? There are three cases to consider. In each case call the unsafe pair x/y, with x the smaller number, and write both numbers in Fibonacci notation.

In the first case x is a B number. Move to reduce y to the number equal to the number obtained by deleting the right-hand digit of x. For example, $x/y = 10/15 = 100100/1000100$. Since 100100 has the 1 farthest to the right at an odd position, it is a B number. Delete its last digit to obtain $10010 = 6$. The safe numbers you must produce (by removing from the larger pile) are 10 and 6. On a chessboard this corresponds to an orthogonal queen move.

In the second case x is an A number, but y exceeds the number obtained by appending 0 to x. Move to reduce the value of y to that number, for example, $x/y = 9/20 = 100010/1010100$. Because x's 1 farthest to the right is in an even position, it is an A number. Appending 0 produces $1000100 = 15$. This is less than 20. Therefore the safe pair to play to is 9/15. On the chessboard this too is an orthogonal queen move.

If the numbers do not conform to cases 1 and 2, do the following:

1. Find the positive difference between x and y.
2. Subtract 1, express the result in Fibonacci notation and change the last digit to 1.
3. Append 0 to get one number. Append two 0's to get a second number. These two numbers are the safe pair you seek, even though the resulting Fibonacci numbers may be "noncanonical" in having consecutive 1's.

An example of the third case is $x/y = 24/32 = 10001000/10101000$. The first and second cases do not apply. The difference between 24 and 32 is 8. Subtracting 1 leaves 7. In Fibonacci notation 7 is 10100. Changing the last digit to 1 produces 10101. Appending 0 and 00 yields the safe pair 101010/1010100 = 12/20. This result is reached by taking 12 from both piles. It corresponds to a diagonal queen move.

It is impossible to go into the whys of Silber's bizarre strategy. Interested readers will find the proofs in Silber's 1977 paper, "Wythoff's Nim and Fibonacci Representations." Neither can I go into the ways in which Wythoff's game has been generalized, but a word or two should be added about the game's reverse, or misère, form: the last person to play loses. As T. H. O'Beirne makes clear in *Puzzles and Paradoxes*, misère Wythoff's Nim, like misère Nim, requires only a trivial alteration of the chart of safe pairs. Remove the first pair, 1/2, and substitute 0/1 and 2/2. The misère strategy is exactly like the standard strategy except that at the end you may have to play to 2/2 or 0/1 instead of 1/2.

Let us modify Wythoff's Nim as follows. A player may take any positive number of counters from either pile, or he may take one counter from one pile and two counters from the other. Can the reader determine the chessboard model and the winning strategy?

ANSWERS

The task was to analyze a game (similar to Nim) in which players may take from either of two piles or take one counter from one pile and two counters from the other. The last person to play wins. In the unbounded-chessboard model explained in the chapter, the first rule is equivalent to the move of a rook west or south and the second rule is equivalent to a knight jumping southwest. The take-away game is therefore isomorphic with the game of cornering a chess piece that combines the powers of rook and knight. Among enthusiasts of unorthodox, or "fairy," chess such a piece is sometimes called a "chancelor" or sometimes an "empress."

If the piece moves only like a rook, the game on the chessboard is the same as standard Nim with two piles. Safe pairs are any two equal positive integers. They correspond to cells on the board's main diagonal that passes through corner cells 0/0 and 7/7. The player who places the rook (on the top row or the column farthest to the right) wins only by putting it on 7/7. Thereafter his strategy is always to move to the diagonal. In the take-away game this means keeping the piles equal. The safe pairs are simply 1/1, 2/2, 3/3. . . .

Surprisingly, giving the rook the additional power of a knight has no effect on this strategy. Applying the recursive technique explained earlier, we find that the safe cells (or safe pairs) are exactly the same as in the rook game.

The misère form of rook-knight Nim (the last person to play loses) is more interesting. The safe pairs are 0/1, 2/3, 4/5, 6/7. . . . On a chessboard these unordered pairs are the cells shown in gray in Figure 52. The "placer" has the win, but he must put the rook-knight on a cell adjacent to the cell in the top right corner. Thereafter he moves to occupy a safe cell. This procedure eventually brings him to 0/1 or 1/0, forcing his opponent to make the final move.

Readers might enjoy analyzing the game on a standard chessboard when the placed piece has other chess powers, in each case limiting moves to west, south and southwest. A "superqueen" or "amazon" (combining queen and knight) means a loss for the placer in standard and reverse play. A king loses for the placer in standard play but wins in misère. The same result emerges if the piece is a king-knight or a king-rook. The placer wins in both types of play if the piece is a king-bishop.

ADDENDUM

Figure 53 shows in gray the safe cells for the king, rook and bishop Nim. The bishop game is trivialized by the fact that the bishop cannot legally move to the target from any square off the main diagonal. If we restrict the bishop to this diagonal, the second player obviously wins the standard game and loses the reverse game.

We can ignore combining the powers of queen and king, queen-bishop or queen-rook or combining rook and bishop, because such

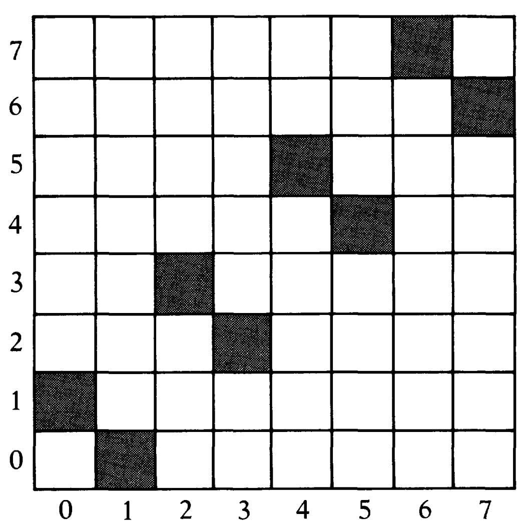


Figure 52
Safe cells of reverse rook-knight Nim

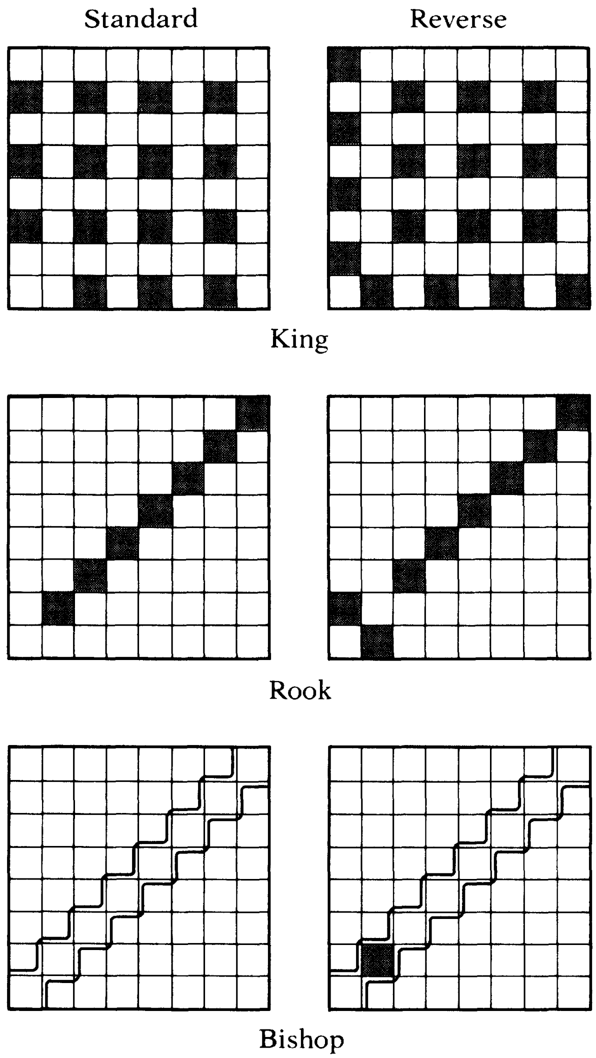


Figure 53 *King, rook and bishop Nim*

pieces are clearly equivalent to the queen. Safe cells are shown in Figure 54 for the superqueen or amazon (queen-knight), superking (king-knight), king-rook (in *shogi*, or Japanese chess, there is such a piece called the *rya-ou*) and the king-bishop (*ryu-ma* in *shogi*). In all cases we assume that a piece can move only west, south or southwest.

Combining bishop and knight produces a piece known to some fairy-chess buffs as the "abbot," to others as the "princess." Christopher Arata sent a detailed analysis of Nimlike games to be played, under various rules, with this piece on an unlimited board. If we limit the playing field to those cells from which it is possible to move to the target square, the top of Figure 55 shows the safe cells for standard and reverse play. Arata suggested allowing the abbot to move southeast as well as

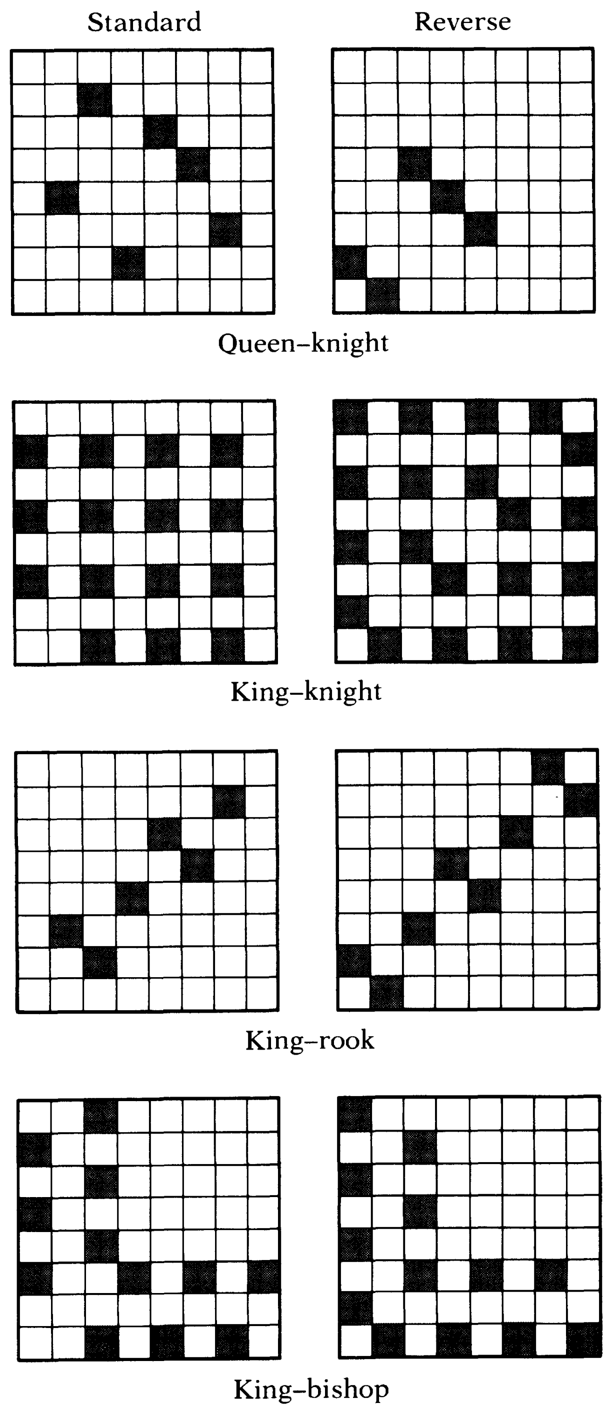


Figure 54 *Pieces with combined powers*

southwest, in which case the safe cells become those shown at the bottom of the illustration. It is not known how to state rules that generalize these patterns to unlimited boards.

All these games have, of course, corresponding rules for playing Nim

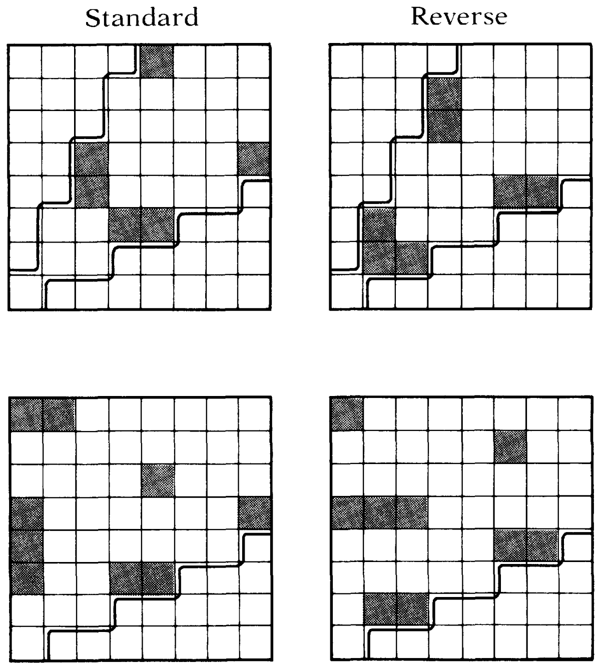


Figure 55 *The abbot (bishop-knight) with S, W and SW moves* (top), *and SE added* (bottom)

with two piles of counters. For other ways of modifying Wythoff's game, readers are referred to papers listed in the bibliography. Many readers pointed out ways in which Silber's algorithm for calculating the winning strategy in Wythoff's Nim can be simplified for efficient computer programs.

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