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NIM AND HACKENBUSH

"The good humour is to steal. . . ."

—WILLIAM SHAKESPEARE, Corporal Nym in *The Merry Wives of Windsor*

In recent decades a great deal of significant theoretical work has been done on a type of two-person game that so far has no agreed-on name. Sometimes these games are called "nim-like games," "take-away games" or "disjunctive games." All begin with a finite set of elements that can be almost anything: counters, pebbles, empty cells of a board, lines on a graph, and so on. Players alternately remove a positive number of these elements in accordance with the game's rules. Since the elements diminish in number with each move, the game must eventually end. None of the moves is dictated by chance; there is "complete information" in that each player knows what his opponent does. Usually the last player to move wins.

The game must also be "impartial." This means that permissible moves depend solely on the pattern of elements prior to the move and not on who plays or on what the preceding moves were. A game in which each player has his own subset of the elements is not impartial. Chess, for example, is partial because a player is not allowed to move an opponent's piece. It follows from the above conditions that every pattern of elements is a certain win for either the first or the second player if the game is played rationally. A pattern is called "safe" (or some equivalent term) if the person who plays next is the loser and "unsafe" if the person who plays next is the winner. Every unsafe pattern can be made safe by at least one move, and every safe pattern becomes unsafe through *any* move. Otherwise it is easy to prove the contradictory result that both players could force a win. The winner's strategy is playing so that every unsafe position left by the loser becomes a safe one.

The best-known example of such a game is nim. The word was coined by the Harvard mathematician Charles L. Bouton when he published the first analysis of the game in 1901. He did not explain why he chose the name, so we can only guess at its origin. Did he have in mind the German *nimm* (the imperative of *nehmen,* "to take") or the archaic English "nim" ("take"), which became a slang word for "steal"? A letter to *The New Scientist* pointed out that John Gay's *Beggar's Opera* of 1727 speaks of a snuffbox "nimm'd by Filch," and that Shakespeare probably had "nim" in mind when he named one of Falstaff's thieving attendants Corporal Nym. Others have noticed that NIM becomes WIN when it is inverted.

Nim begins with any number of piles (or rows) of objects with an arbitrary number in each pile. A move consists in taking away as many objects as one wishes, but only from one pile. At least one object must be taken, and it is permissible to take the entire pile. The player who takes the last object wins. Bouton's method of determining whether a nim position is safe or unsafe is to express the pile numbers in binary notation, then add them without carrying. If and only if each column adds to an even number (zero is even) is the pattern safe. An equivalent but much easier way to identify the pattern (with practice one can do it in one's head) is to express each pile number as a sum of distinct powers of 2, eliminate all pairs of like powers and add the powers that remain. The final sum is the nim sum of the pattern. In current parlance this is called the "Grundy number" or "Sprague-Grundy" number of the pattern, after Roland Sprague and P. M. Grundy, who independently worked out a general theory of take-away games based on assigning (by techniques that vary with different games) single numbers to each state of the game.

For example, assume that a game of nim begins with three piles of three, five and seven counters.

$$3 = \cancel{2} + \cancel{1}$$
$$5 = \cancel{4} + \cancel{1}$$
$$7 = \cancel{4} + \cancel{2} + 1$$

Pairs of 4's, 2's and 1's are crossed out as shown. The sum of what remains is 1. This is the nim sum of the pattern. If and only if the nim sum is zero is the pattern safe, otherwise it is unsafe (as it is here). If you play an unsafe pattern, you win by changing it to safe. Here removing one counter from any pile will lower the nim sum to zero. In three-pile nim, with no pile exceeding seven counters, the safe nim patterns are 0–*n*–*n*, where *n* in the first triplet is any digit from 1 through 7, and 1–2–3, 1–4–5, 1–6–7, 2–4–6, 2–5–7, 3–4–7, 3–5–6. If your opponent plays next, he is sure to leave a pattern with a nonzero nim sum that you can lower to zero again, thereby maintaining your winning strategy.

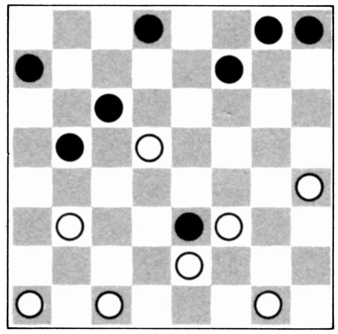
Like all games of this type, nim has its *misère* form, in which the player who takes the last piece is the loser. In many take-away games the strategy of *misère* play is enormously complicated, but in nim only a trivial modification is required at the end of the play. The winner need only play a normal strategy until it is possible to leave an odd number of single-counter piles. This forces his opponent to take the last counter.

Many take-away games seem to demand a strategy different from that of nim but actually do not. Suppose the rules of nim allow a player (if he wishes) to take from a pile, then divide the remaining counters of that pile into two separate piles. (If the counters are in rows, this is the same as taking contiguous counters from inside a row and regarding those that remain as being two distinct rows.) One might expect this maneuver to complicate the strategy, but it has no effect whatever. To win, compute the nim sum of a position in the usual way and, if it is unsafe, play a standard move to make it safe. For example, in the 3–5–7 game suppose your first move is taking a counter from the three-pile, leaving the safe 2–5–7. Your opponent removes two counters from the seven-pile and splits the remaining five counters into a two-pile and a three-pile. The pattern is now 2–5–2–3. Its nim sum is six, which you make safe by taking two from the five-pile.

A pleasant counter-moving game on a chessboard is shown in Figure 83. No fewer than two columns may be used. In this example we use all eight columns. Black and white counters are placed on arbitrary squares in each column, black on one side, white on the other. (A randomizing device, such as a die, can be used for the placement.) Players sit on opposite sides and alternate moves. A move consists in advancing one of your counters any desired number of empty cells in its column. It may not leap its opposing counter, so that when two counters meet, neither may move again. The last player to move wins.

An astute reader may see at once that this game is no more than a thinly disguised nim. The "piles" are the empty cells between each pair of opposing counters. In the illustration, the piles are 5–1–4–2–0–3–6–3, which has an unsafe nim sum of

Figure 83



A nim game on a chessboard

4. The first player can win by moving the counter in column one, three or seven forward four spaces. If the game had begun with all the counters in each player's first row, the pattern would have been 6–6–6–6–6–6–6–6, a safe position because its nim sum is zero. The first player must lose. The second player groups the columns into four pairs, then duplicates each of his opponent's moves in the paired column, a strategy that ensures a zero nim sum after every move.

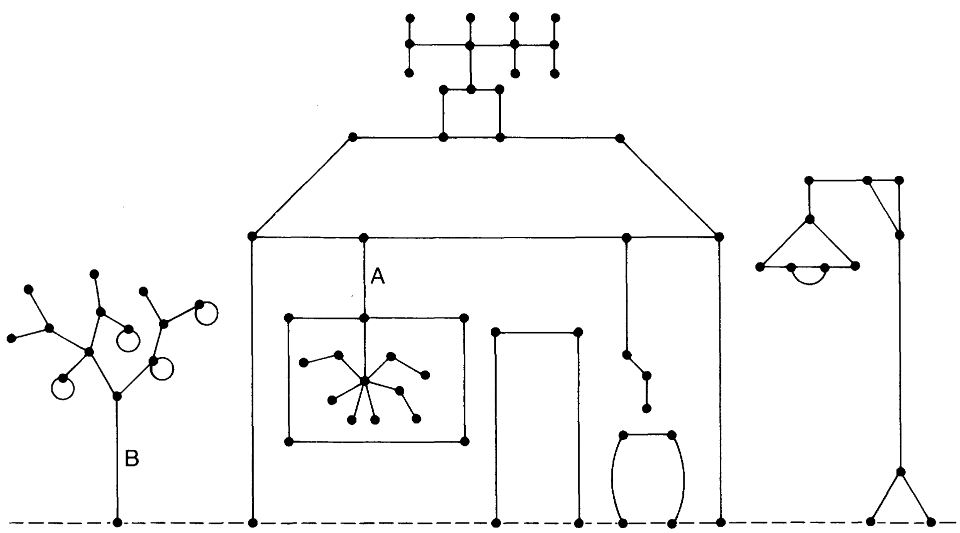
Suppose we complicate the rules by allowing either player to move backward as well as forward. Such a retreat is equivalent, of course, to *adding* counters to a nim pile. How does this affect the winning strategy?

A better-disguised game based on nim addition is a delightful pencil-and-paper game recently invented by John Horton Conway, the University of Cambridge mathematician who invented "Life," the topic of three of this book's chapters. Conway calls the new game Hackenbush, but it has also been called Graph and Chopper, Lizzie Borden's Nim and other names.

The initial pattern is a set of disconnected graphs, such as the Hackenbush Homestead as drawn by Conway [*see Figure* 84]. An "edge" is any line joining two "nodes" (spots) or one node to itself. In the latter case the edge is a "loop" (for example, each apple on the tree). Between two nodes there can be multiple edges (for example, the light bulb). Every graph stands on a base line that is not part of the graph. Nodes on the base line, which is shown as a broken line in the illustrations, are called "base nodes."

Two players alternate in removing any single edge. Gravity now enters the game because taking an edge also removes any portion of the graph that is no longer connected to the base line. For instance, removing edge *A* eliminates both the spider

Figure 84



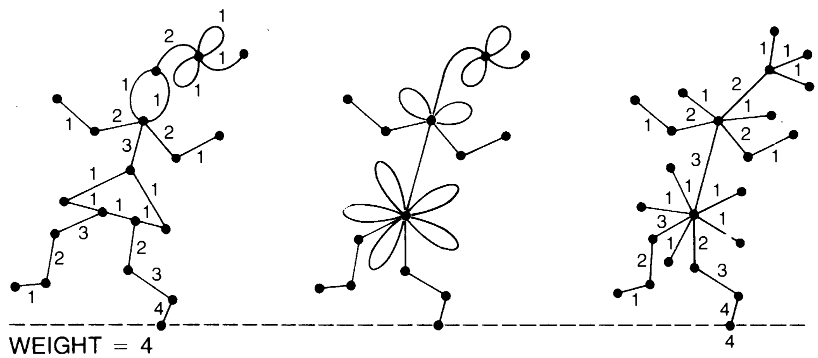
The Hackenbush Homestead

and the window since both would fall to the ground, but removing the edge joining the spider to the window removes only the spider. Taking edge *B* chops down the entire apple tree. If one edge of the streetlight's base is taken, the structure still stands, but taking the second edge on a later move topples the entire structure. The person who takes the picture's last edge is the winner.

As in nim, every picture is either safe (second-player win) or unsafe (first-player win), and the winner's strategy is to convert every unsafe pattern to safe. To evaluate a picture each graph must be assigned a number measuring the graph's "weight." To arrive at the assignment the first step is to collapse all the "cycles" (closed circuits of two or more edges) to loops, turning the graph into what Conway calls an apple tree, although in many cases the loops are best regarded as being flower petals. To see how it works, consider Conway's girl [*see Figure* 85]. She incorporates two cycles: her head and her skirt. First the two nodes of her head are brought together and then the two edges are bent into loops. Do the same with the five nodes and five edges of the skirt. The girl is now a flower girl [*middle figure*]. The next step is to change her to an ordinary tree by replacing each loop with a single branch [*figure at right*].

We now calculate this tree's weight. First, label 1 all edges with a terminal node (a node unconnected to another edge) or, to put it differently, all edges that, if removed, cause no other edges to fall off the tree. Label 2 all edges that support only

Figure 85



Girl on one foot

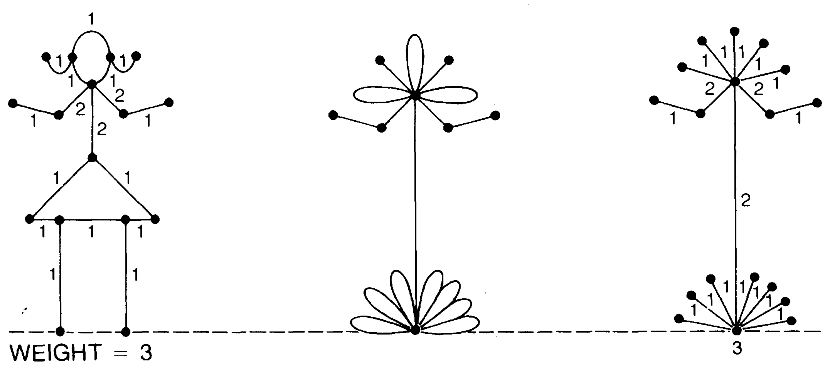
one edge. Each remaining edge is labeled with one more than the nim sum of all the edges it immediately supports. Consider the edge corresponding to the girl's hair between her head and her hair ribbon. It immediately supports 1–1–1. A pair of 1's cancel, giving a nim sum of 1. Add 1 to the nim sum and this edge has a weight of 2. The edge that forms the body above the skirt immediately supports edges of values 2–1–2–1–2. The nim sum is 2. Add 1 and the edge has a weight of 3.

The girl's unraised thigh supports 3–1–1–3–1–1–1, a nim sum of 1, to which 1 is added to give the thigh a value of 2. The calf below it has a value of 3, the foot a value of 4. (In each case we simply add 1 to the value of the single, immediately supported edge.) Since the foot is the only support of the entire graph, the girl has a weight of 4. All edge values are now transferred to corresponding edges on the original girl.

With practice, edge values can be computed directly on the original graph, but it requires great care. For example, the girl's five skirt edges, raised thigh, and body are all "immediately" supported by her unraised thigh. This is clear in the tree graph but is not so obvious in the original graph because many of the immediately supported edges are not close to the thigh.

If a graph has more than one base node, such as the door, barrel and lamp in the Homestead, collapse the base cycle into loops, remembering that the broken line segment between a pair of base nodes is not part of the graph. The door's transformations are shown in Figure 87b. Since the nim sum of 1–1–1 is 1, the door's weight is 1. A girl standing on both feet [*see Figure* 86] has a weight of 3. Note how the two cycles formed by her skirt and legs collapse into seven loops. A winning move, for a game played with her alone, is taking the top of her head or one of her hairs. This lowers the value of her

Figure 86



Girl on both feet

head to zero, her body to 1 and her weight to zero. In this manner a weight can be assigned to each of the five graphs that make up the Hackenbush Homestead: The apple tree, house (including window, spider, chimney, television antenna and drainpipe), door, barrel and streetlight.

If Hackenbush is played with only the girl on one foot, the game is as trivial as playing nim with only one pile. The first player can win at once by taking the supporting foot. The poor girl collapses and he acquires all her edges. In the case of a figure with more than one base node, such as the door, we must remember to take an edge so that the remaining nim sum is zero. A first player can do this only by taking the door's top edge, leaving two graphs of weight 1 each, or a combined nim sum of zero. Taking either side leaves only one graph (of weight 2), which can be taken entirely by the second player.

A picture consisting of n graphs, such as the five graphs of the Homestead, is treated exactly like five piles in nim. The nim sum of all the weights is the total Grundy number. If and only if this number is zero is the picture safe and the second player assured of winning. As in nim, the winning strategy is to play so that the nim sum of what remains is always zero.

The reader is invited to determine the weight of each graph in the Hackenbush Homestead and verify that the Homestead's nim sum is 10. Since this is not zero, the first player can win. It turns out (of course Conway designed it that way) that there is only one edge the first player can take that will guarantee a win by lowering the nim sum to zero. Which edge is it?

My account of hackenbush is only a brief introduction to this game. For a fantastic amount of additional information about the game, its deep theorems and its numerous variations, see

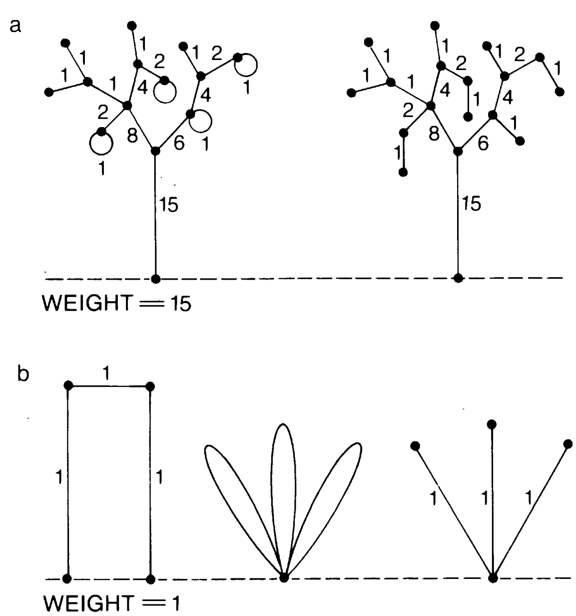
Conway's *On Numbers and Games,* and the two volumes of *Winning Ways* by Berlekamp, Conway, and Guy. Both works also contain an abundance of material on other nim-like games and the theory behind them in both standard and *misère* play.

ANSWERS

The first problem was to explain how the winning strategy in a chessboard version of nim is affected by allowing players to move their counters backward. The answer: It has almost no effect. If the loser retreats, the winner merely advances his opposing counter until the number of spaces separating the two men is the same as before. This preserves the status quo, leaving the basic strategy unaltered. The winner never retreats and, since the chessboard is finite, the loser's retreats must eventually cease. This variation of the game has been attributed to D. G. Northcott and is known as Northcott's nim.

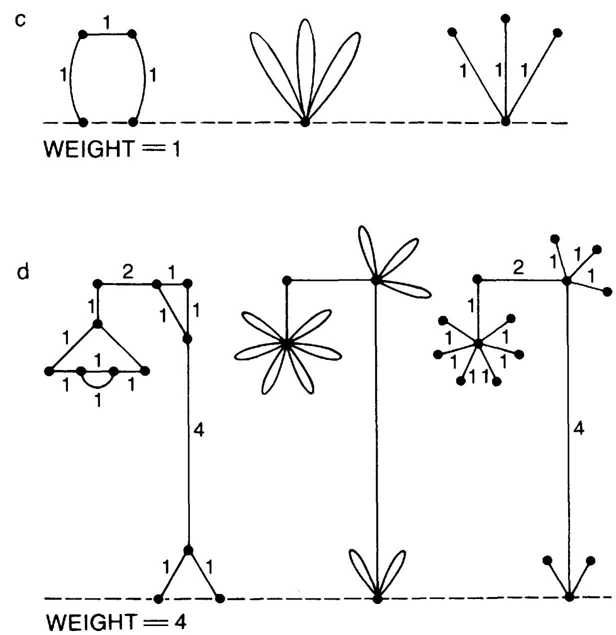
How the various parts (graphs) of John Horton Conway's Hackenbush Homestead are transformed, as explained, into apple trees, then trees and labeled is shown in Figures 87, 88. The graphs have weights of 15–1–1–4–1, therefore the Homestead's nim sum is 10. The only way the first player can reduce this Grundy number to zero is by lowering the apple tree's weight to 5. "The tree trunk supports two branches of 8 and 6," Conway writes, "and these must be changed to 2 and 6, or

Figure 87 a & b



Weighing the Hackenbush apple tree, door, barrel and streetlight

Figure 87 c & d



Weighing the Hackenbush apple tree, door, barrel and streetlight

8 and 12, to have nim sum 4. Clearly we must choose the left branch. Climbing the tree, we discover that there is a unique winning move—chop the twig bearing the second apple from the left."

This chop lowers the tree's weight (the value of its trunk) to 5 [*see Figure* 89]. The graphs now have weights of 5–1–1–4–1, which have a nim sum of zero.

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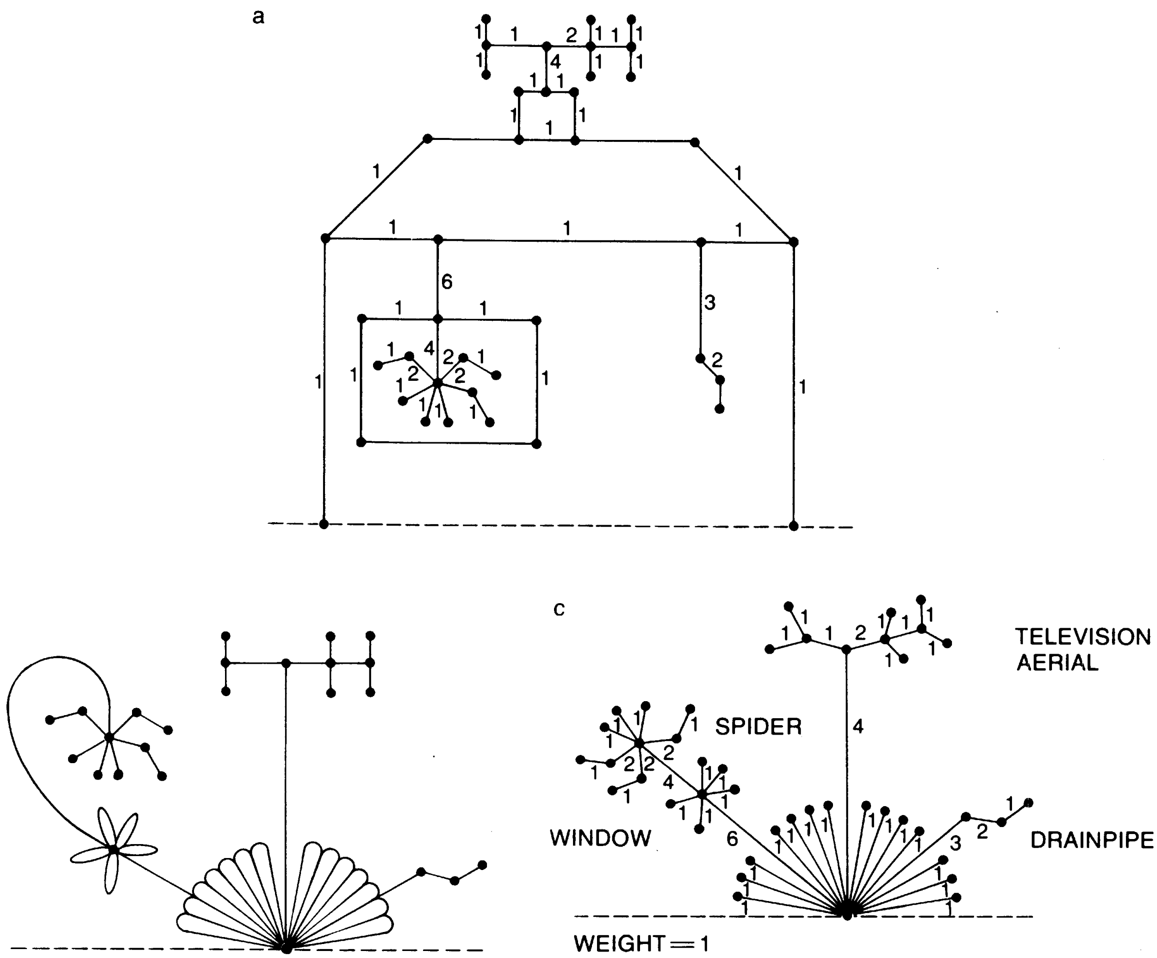
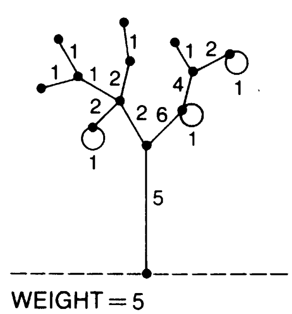
Figure 88



Weighing the Hackenbush house

Figure 89



Apple tree after the winning chop

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