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Generalized Ticktacktoe

The world's simplest, oldest and most popular pencil-and-paper game is still ticktacktoe, and combinatorial mathematicians, often with the aid of computers, continue to explore unusual variations and generalizations of it. In one variant that goes back to ancient times the two players are each given three counters, and they take turns first placing them on the three-by-three board and then moving them from cell to cell until one player gets his three counters in a row. (I discuss this game in my *Scientific American Book of Mathematical Games and Diversions.*) Moving-counter ticktacktoe is the basis for a number of modern commercial games, such as John Scarne's Teeko and a new game called Touché, in which concealed magnets cause counters to flip over and become opponent pieces.

Standard ticktacktoe can obviously be generalized to larger fields. For example, the old Japanese game of go-moku ("five stones") is essentially five-in-a-row ticktacktoe played on a go board. Another way to generalize the game is to play it on "boards" of three or more dimensions. These variants and others are discussed in my *Wheels, Life* book.

In March, 1977, Frank Harary devised a delightful new way to generalize ticktacktoe. Harary was then a mathematician at the University of Michigan. He is now the Distinguished Professor of Computer Science at New Mexico State University, in Las Cruces. He has been called Mr. Graph Theory because of his tireless, pioneering work in this rapidly growing field that is partly combinatorial and partly topological. Harary is the founder of the *Journal of Combinatorial Theory* and the *Journal of Graph Theory,* and the author of *Graph Theory,* considered the world over to be the definitive textbook on the subject. His papers on graph theory, written alone or in collaboration with others, number more than 500. Harary ticktacktoe, as I originally called his generalization of the game, opens up numerous fascinating areas of recreational mathematics. Acting on his emphatic request, I now call it animal ticktacktoe for reasons we shall see below.

We begin by observing that standard ticktacktoe can be viewed as a two-color geometric-graph game of the type Harary calls an achievement game. Replace the nine cells of the ticktacktoe board with nine points joined by lines, as is shown in Figure 90. The players are each assigned a color, and they take turns coloring points on the graph. The first player to complete a straight line of three points in his color wins. This game is clearly isomorphic with standard ticktacktoe. It is well known to end in a draw if both players make the best possible moves.

Let us now ask: What is the smallest square on which the first player can force a win by coloring a straight (non-diagonal) three-point path? It is easy to show that it is a square of side four. Harary calls this side length the board number b of the game. It is closely related to the Ramsey number of generalized Ramsey graph theory, a number that plays an important part in the Ramsey games. (Ramsey theory is a field in which Harary has made notable contributions. It was in a 1972 survey paper on Ramsey theory that Harary first proposed making a general study of games played on graphs by coloring the graph edges.) Once we have determined the value of b we

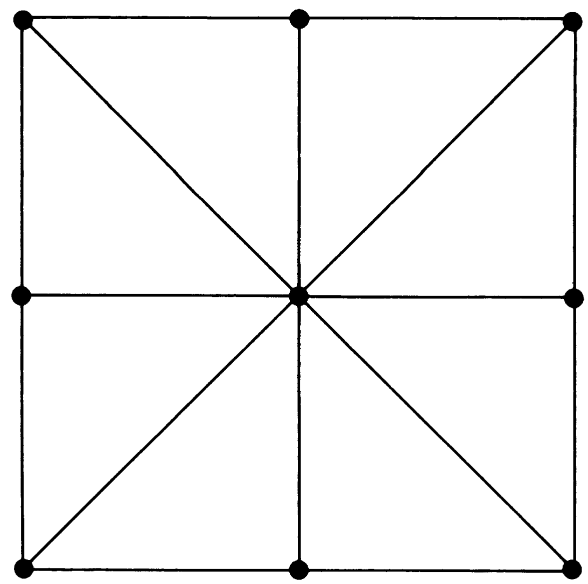


FIGURE 90 Ticktacktoe as a two-coloring game

can ask a second question. In how few moves can the first player win? A little doodling shows that on a board of side four the first player can force a win in only three moves. Harary calls this the move number *m* of the game.

In ticktacktoe a player wins by taking cells that form a straight, order-3 polyomino that is either edge- or corner-connected. (The corner-connected figure corresponds to taking three cells on a diagonal.) Polyominoes of orders 1 through 5 are depicted in Figures 91 and 92. The polyomino terminology was coined by Solomon W. Golomb, who was the first to make a detailed study of these figures. Harary prefers to follow the usage of a number of early papers on the subject and call them "animals." I shall follow that practice here.

We are now prepared to explain Harary's fortuitous generalization. Choose an animal of any order (number of square cells) and declare its formation to be the objective of a ticktacktoelike game. As in ticktacktoe we shall play not by coloring spots on a graph but by marking cells on square matrixes with noughts and crosses in the usual manner or by coloring cells red and green as one colors edges in a Ramsey graph game. Each player tries to label or color cells that will form the desired animal. The animal will be accepted in any orientation and, if it is asymmetrical, in either of its mirror-image forms.

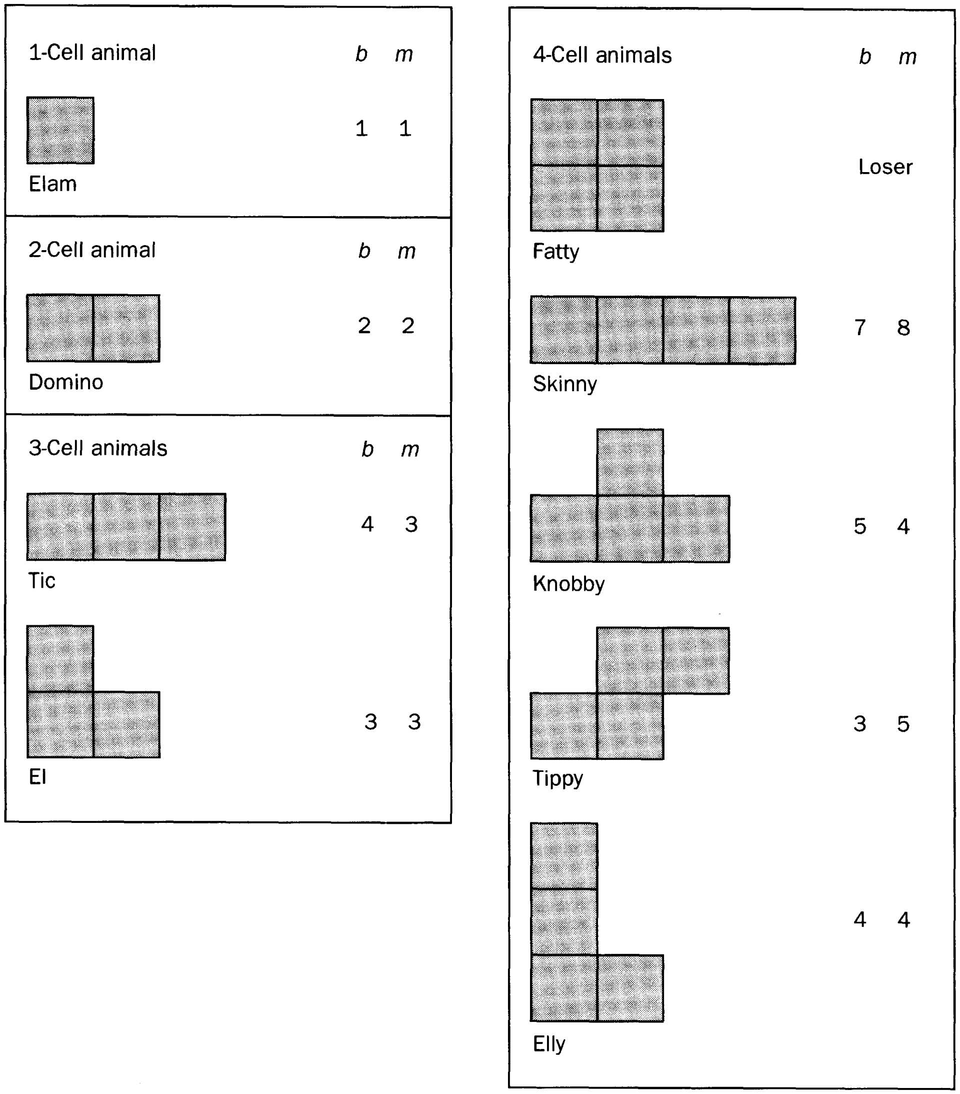


FIGURE 91 Animals of 1 cell through 4 cells

Our first task is to determine the animal's board number, that is, the length of the side of the smallest square on which the first player can, by playing the best possible strategy, force a win. If such a number exists, the animal is called a winner, and it will be a winner on all larger square fields. If there is no board number, the animal is called a loser. If the animal chosen as the objective of a game is a loser, the second player can always force a draw, but he can never force a win. The clever proof of this fact is well known and applies to most ticktacktoelike games. Assume that the second player has a

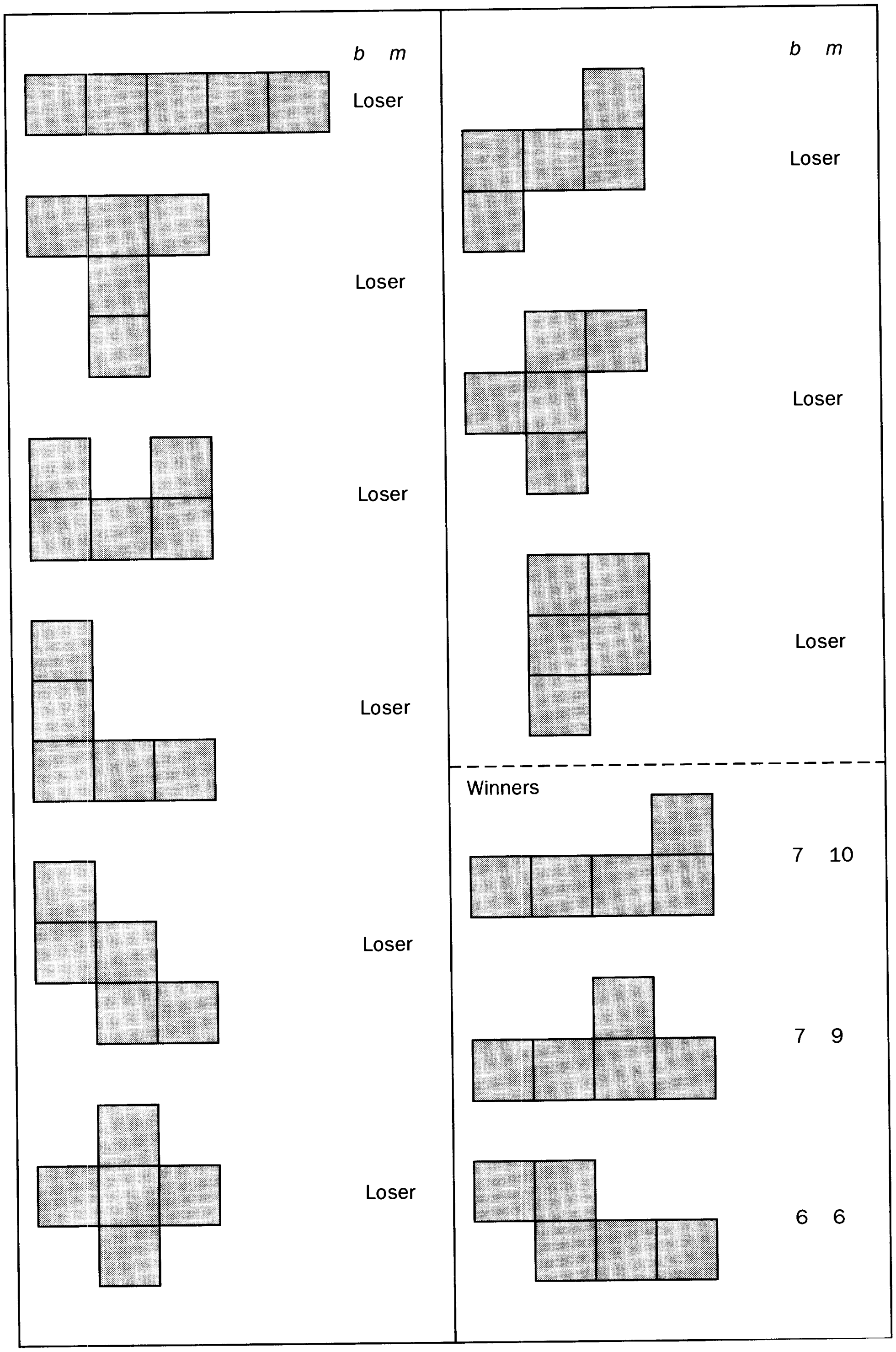


FIGURE 92 The 12 animals of 5 cells

winning strategy. The first player can "steal" the strategy by first making an irrelevant opening move (which can never be a liability) and thereafter playing the winning strategy. This finding contradicts the assumption that the second player has a winning strategy, and so that assumption must be false. Hence the second player can never force a win. If the animal is a winner and b is known, we next seek m, the minimum number of moves in which the game can be won.

For the 1-cell animal (the monomino), which is trivially a winner, b and m are both equal to 1. When, as in this case, m is equal to the number of cells in the animal, Harary calls the game economical, because a player can win it without having to take any cell that is not part of the animal. The game in which the objective is the only 2-cell animal (the domino) is almost as trivial. It is also economical, with b and m both equal to 2. The games played with the two 3-cell animals (the trominoes) are slightly more difficult to analyze, but the reader can easily demonstrate that both are economical: for the L-shaped 3-cell animal b and m are both equal to 3, and for the straight 3-cell animal b equals 4 and m equals 3. This last game is identical with standard ticktacktoe except that corner-connected, or diagonal, rows of three cells are not counted as wins.

It is when we turn to the 4-cell animals (the tetrominoes) that the project really becomes interesting. Harary has given each of the five order-4 animals names, as is shown in Figure 91. Readers may enjoy proving that the b and m numbers given in the illustration are correct. Note that Fatty (the square tetromino) has no such numbers and so is labeled a loser. It was Andreas R. Blass, one of Harary's colleagues at Michigan, who proved that the first player cannot force Fatty on a field of any size, even on the infinite lattice. Blass's result was the first surprise of the investigation into animal ticktacktoe. From this finding it follows at once that any larger animal containing a two-by-two square also is a loser: the second player simply plays to prevent Fatty's formation. More generally, any animal that contains a loser of a lower order is itself a loser. Harary calls a loser that contains no loser of lower order a basic loser. Fatty is the smallest basic loser.

The proof that Fatty is a minimal loser is so simple and elegant that it can be explained quickly. Imagine the infinite plane tiled with dominoes in the manner shown at the top of Figure 93. If Fatty is drawn anywhere on this tiling, it must

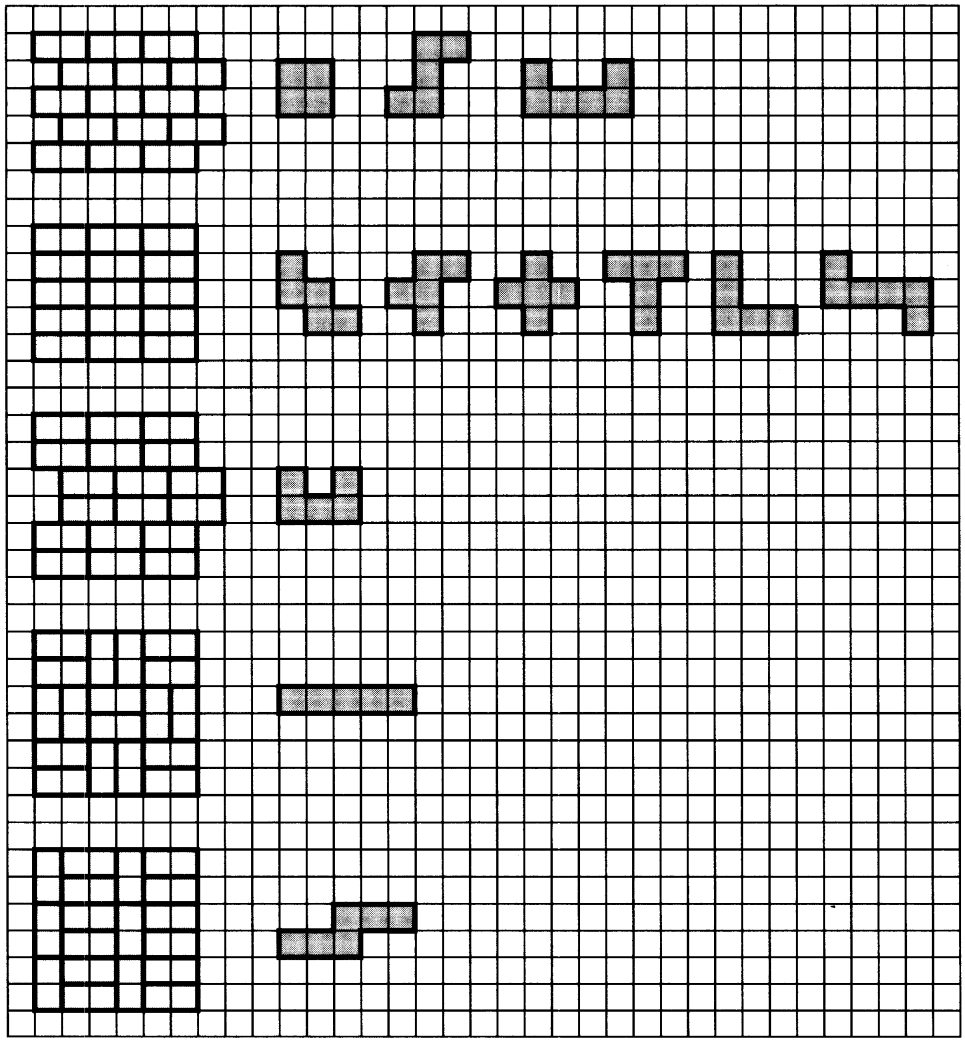


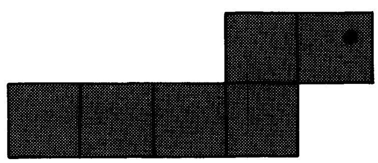
FIGURE 93 Tiling patterns (left) for the 12 basic losers (right)

contain a domino. Hence the second player's strategy is simply to respond to each of his opponent's moves by taking the other cell of the same domino. As a result the first player will never be able to complete a domino, and so he will never be able to complete a Fatty. If an animal is a loser on the infinite board, it is a loser on all finite boards. Therefore Fatty is always a loser regardless of the board size.

Early in 1978 Harary and his colleagues, working with only the top four domino tilings shown in Figure 93 established that all but three of the 12 5-cell animals are losers. Among the nine losers only the one containing Fatty is not a basic loser. Turning to the 35 6-cell animals, all but four contain basic losers of lower order. Of the remaining four possible winners three can be proved losers with one of the five tilings shown in the illustration. The animals that can be proved basic losers with each tiling pattern are shown alongside the pattern. In every case the proof is the same: it is impossible to draw the loser on the associated tiling pattern (which is assumed to be infinite) without including a domino; therefore the second player can always prevent the first player from forming the animal by following the strategy already described for blocking Fatty. There are a total of 12 basic losers of order six or lower.

It is worth noting how the tiling proof that the straight animal of five cells is a loser (another proof that was first found by Blass) bears on the game of go-moku. If the game is limited to an objective of five adjacent cells in a horizontal or vertical line (eliminating wins by diagonal lines), the second player can always force a draw. When diagonal wins are allowed, the game is believed to be a first-player win, although as far as I know that has not yet been proved even for fields larger than the go board.

The only 6-cell animal that may be a winner is the one that I named Snaky:



Although they have not yet been able to prove this animal is a winner, they conjecture its board number, if any, is no larger than 15 and its move number is no larger than 13. This assertion is the outstanding unsolved problem in animal ticktacktoe theory. Perhaps a reader can prove Snaky is a loser or conversely show how the first player can force the animal on a square field and determine its board and move numbers.

All the 107 order-7 animals are known to be losers because each contains a basic loser. Therefore since every higher-order animal must contain an order-7 animal, it can be said with confidence that there are no winners beyond order 6. If Snaky is a winner, as Harary and his former doctoral student Geoffrey Exoo conjecture, there are, by coincidence, exactly a dozen winners—half of them economical—and a dozen basic losers.

Any 4- or 5-cell animal can be the basis of a pleasant pencil-and-paper game or a board game. If both players know the full analysis, then depending on the animal chosen either the first player will win or the second player will force a draw. As in ticktacktoe between inexpert players, however, if this knowledge is lacking, the game can be entertaining. If the animal chosen as the objective of the game is a winner, the game is best played on a board of side b or $b-1$. (Remember that a square of side $b-1$ is the largest board on which the first player cannot force a win.)

All the variations and generalizations of animal ticktacktoe that have been considered so far are, as Harary once put it, "Ramseyish." For example, one can play the misère, or reverse, form of any game—in Harary's terminology an avoidance game—in which a player wins by forcing his opponent to color the chosen animal.

Avoidance games are unusually difficult to analyze. The second player trivially wins if the animal to be avoided is the monomino. If the domino is to be avoided, the second player obviously wins on the 2×2 square, and almost as obviously on the 2×3 rectangle.

On a square board of any size the first player can be forced to complete the L-shaped 3-cell animal. Obviously the length of the square's side must be at least 3 for the game to be meaningful. If the length of the side is odd, the second player will win if he follows each of his opponent's moves by taking the cell symmetrically opposite the move with respect to the center of the board. If the first player avoids taking the center, he will be forced to take it on his last move and so will lose. If he takes it earlier in the game without losing, the second player should follow with any safe move. If the first player then takes the cell that is symmetrically opposite the second player's move with respect to the center, the second player should again make a harmless move, and so on; otherwise he should revert to his former strategy. If the length of the square's side is even, this type of symmetrical play leads to a draw, but the second player can still win by applying more complicated tactics.

On square boards the straight 3-cell animal cannot be forced on the first player. The proof of this fact is a bit difficult, even for the three-by-three square, but as a result no larger animal containing the straight 3-cell species can be forced on any square board. (The situation is analogous to that of basic losers in animal-achievement games.) Hence among the 4-cell animals only Fatty and Tippy remain as possible nondraws. Fatty can be shown to be a draw on any square board, but Tippy can be forced on the first player on all square boards of odd side. The complete analysis of all animal-avoidance games is still in the early stages and appears to present difficult problems.

Harary has proposed many other nontrivial variants of the basic animal games. For example, the objective of a game can be two or more different animals. In this case the first player can try to form one animal and the second player the other, or both players can try to form either one. In addition, achievement and avoidance can be combined in the same game, and nonrectangular boards can be used. It is possible to include three or more players in any game, but this twist introduces coalition play and leads to enormous complexities. The rules can also be revised to accept corner-connected animals or animals that are both edge- and corner-connected. At the limit, of course, one could make any pattern whatsoever the objective of a ticktacktoelike game, but such broad generalizations usually lead to games that are too complicated to be interesting.

Another way of generalizing these games is to play them with polyiamonds (identical edge-joined equilateral triangles) or polyhexes (identical edge-joined regular hexagons) respectively on a regular triangular field or a regular hexagonal field. One could also investigate games played with these animals on less regular fields. An initial investigation of triangular forms, by Harary and Heiko Harborth, is listed in the bibliography.

The games played with square animals can obviously be extended to boards of three or more dimensions. For example, the 3-space analogue of the polyomino is the polycube: n unit cubes joined along faces. Given a polycube, one could seek b and m numbers based on the smallest cubical lattice within which the first player can force a win and try to find all the polycubes that are basic losers. This generalization is almost totally unexplored, but see the bibliography for a paper on the topic by Harary and Michael Weisbach.

As I have mentioned, Blass, now at Pennsylvania University, is one of Harary's main collaborators. The others include Exoo, A. Kabell and Heiko Harborth, who is investigating games with the triangular and hexagonal cousins of the square animals. Harary is still planning a book on achievement and avoidance games in which all these generalizations of ticktacktoe and many other closely related games will be explored, and he is also persuading his current computer science students to develop computer programs for playing these games both offensively and defensively. This is the area of AI (artificial intelligence) known as game-playing programs.

ADDENDUM

In giving the proof that a second player cannot have the win in most ticktacktoe-like games, I said that if the first player always wins on a board of a certain size, he also wins on any larger board. This is true of the square boards with which Harary was concerned, but is not necessarily true when such games are played on arbitrary graphs. A. K. Austin and C. J. Knight, mathematicians at the University of Sheffield, in England, sent the following counterexample.

Consider the graph at the left of Figure 94, on which three-in-a-row wins. The first player wins by taking *A*. The second player has a choice of taking a point in either the small or the large triangle. Whichever he chooses, the first player takes a corner point in the other triangle. The opponent must block the threatened win, then a play in the remaining corner of the same triangle forces a win.

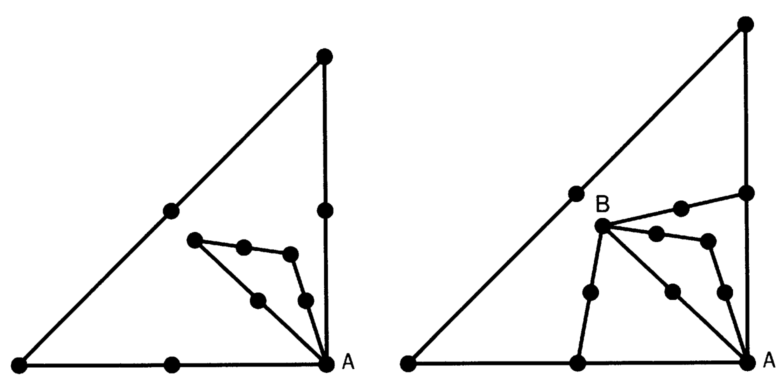


FIGURE 94 First player wins on graph at left, but second player can force a draw on enlarged graph at right

Now enlarge the "board" by adding two points as shown on the right in Figure 94. The second player can draw by playing at *B*. If the first player does not start with *A*, the second player draws by taking *A*.

Achievement and avoidance games played on graphs obviously open up endless possibilities that will be explored in Harary's forthcoming book.

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