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TICKTACKTOE GAMES

> "It's as simple as tit-tat-toe, three-in-a-row, and as easy as playing hooky. I should *hope* we can find a way that's a little more complicated than *that,* Huck Finn."
>
> —Mark Twain,
> *The Adventures of Huckleberry Finn*

Ticktacktoe (the spelling varies widely) is not nearly so simple as Tom Sawyer thought. When Charles Sanders Peirce wrote his *Elements of Mathematics,* a textbook that was not published until 1976, he included a 17-page analysis of only the side opening of this ancient game. It was one of Peirce's many anticipations of "modern math." Today's progressive teachers frequently use ticktaktoe to introduce their pupils to such concepts as the intersection of sets, rotational and mirror-reflection symmetry, and higher Euclidean space. In this chapter we consider some unusual aspects of the game not covered in two earlier columns reprinted in *The Scientific American Book of Mathematical Puzzles & Diversions* (Chapter 4), and *Mathematical Carnival* (Chapter 16).

The traditional game, as most readers surely know, is a draw if both players do their best. From time to time pictures of a ticktacktoe game appear in advertisements and cartoons, and sometimes they provide pleasant little puzzles. For example, on May 13, 1956, in the New York *Herald Tribune,* there was an IBM advertisement with the unfinished game at the left in Figure 52. Which player went first, assuming that the players were not stupid? It takes only a moment to realize that *O* could not

Figure 52

	X	
	O	X
		O

X	O	O
O	X	O
X	O	X

X		X
	O	
O		X

Three easy ticktacktoe puzzles

have gone first or *X* would have played the top left corner on his second move. The other two patterns are almost as trivial. Does the center one, from a Little Lulu cartoon in *The Saturday Evening Post* (January 16, 1937), depict a possible game? At the right is a pattern from an advertisement by publisher Lyle Stuart in *The New York Times* (June 1, 1971). In which cell must the last move have been made?

If the first player, say *X*, opens in the center cell, he can force a draw that always ends with the same basic pattern. This underlies several prediction tricks. For example, the magician draws the finish of a game, with all cells filled, on a square sheet of paper that he turns face down without letting anyone see it. He then plays a ticktacktoe game with someone, writing on another square sheet. After the game ends in a draw he turns over his "prediction." The two patterns match cell for cell.

The technique is explained in Figure 53. *X* plays the center opening. If *O* marks any corner cell, *X* forces the draw shown at the left in the illustration (moves are numbered in order of play). It is only necessary for *X* to remember where to make his second move, since all moves are forced from then on; a simple rule for the second move is to consider the corner opposite *O*'s first move and then play adjacent to it on the clockwise side. If *O* responds to the opening with a side cell, *X* forces the draw shown at the right. In this situation only *O*'s moves are forced

Figure 53

O_2	O_4	X_5
X_7	X_1	O_8
O_6	X_3	X_9

O_8	O_2	X_3
X_5	X_1	O_6
O_4	X_9	X_7

A ticktacktoe prediction trick

and X must remember how to play his next four moves. The following simple rule was proposed by a magician who signed himself "Thorson" when he described this trick in the September 1960 issue of *M.U.M.*, official organ of the Society of American Magicians: X makes his second, third and fourth moves adjacent and clockwise to O's previous moves, and his fifth move in the only remaining empty cell.

Note that the two final results are identical. Of course, each game can be played in any of four different orientations. The magician, recalling which corner of his inverted prediction has the O surrounded by three X's, casually turns over the square sheet along the proper axis—orthogonal or diagonal—so that his predicted pattern matches the orientation of the game just completed.

The trick can even be repeated. This time X substitutes counterclockwise for clockwise in the rules, having drawn a prediction that is a mirror image of the preceding one. The two predictions will not match in any orientation and few people will realize that they are mirror reflections of each other.

Dozens of variations of planar ticktacktoe have been analyzed. Standard games on squares of higher order than 3, when the goal on an order-n board is to get n in a row, are uninteresting because the second player can easily force a draw. My first column on ticktacktoe discussed games in which counters are moved over the board (one such version goes back to ancient Greece), and toetacktick, in which the first to get three in a row *loses*.

A. K. Austin's "wild ticktacktoe," in which players may use either X or O on every move, was shown to be a first-player win in my *Sixth Book of Mathematical Games*, Chapter 12, Problem 3. What about "wild toetacktick," in which players can choose either mark on each move and the first three-in-a-row loses? In 1964 Solomon W. Golomb and Robert Abbott independently found that the simple symmetry strategy by which the first player can force at least a draw in standard toetacktick also applies to the wild version. A center opening is followed by playing directly opposite the other player's moves, always choosing X if he played O and O if he played X. The question remains: Does the first player have a *winning* strategy in wild toetacktick? Abbott made an exhaustive tree diagram of all possible plays and proved that the second player too can force a draw. Tame toetacktick also is a draw if both sides play rationally.

An amusing variation appears in David L. Silverman's book of game puzzles, *Your Move*. The rules are the same as in standard ticktacktoe except that one player tries to achieve a draw and the other player wins if either of them gets three in a row. Can the reader show that no matter who plays first the player trying to force a row of three can always do so? Silverman does not answer this in his book, but I shall give his solution in the Answer Section.

It is impossible to describe all the other planar variants that have been proposed, such as using numbers or letters as marks for the goal of forming a certain sum or spelling a certain word; playing on the vertexes of curious nine-point graphs (for a game on one such graph see my *Mathematical Magic Show*, Chapter 5, Problem 5); using counters with *X* on one side and *O* on the other, with the counters turned over according to specified rules. Games have been marketed in which flip-overs are randomized by concealed magnets that may or may not reverse a counter or by tossing beanbags at a board to cause cubical cells to alter their top symbols by rotating.

If ticktacktoe is played on an unlimited checkerboard, it is a trivial win for the first player if the goal is to get four or any smaller number of one's marks in an orthogonal or diagonal row. When the goal is five in a row, this game is far from trivial. It is the ancient Oriental game known as go-moku (five stones) in Japan, where it is played on the intersections of a go board. (The game is sold in the U.S. by Parker Brothers under the name of Pegity.) Although it is widely believed that a first-player winning strategy exists, this has not yet, to my knowledge, been proved.

There is no doubt about the first player's strong advantage in unrestricted go-moku. Indeed, it is so overwhelming that in Japan the standard practice is to weaken the first player by not allowing the following moves:

(1) A move that simultaneously creates a "fork" of two or more intersecting rows of open threes. By "open three" is meant any pattern in which a play will form a row of four adjacent stones that is open at both ends. There is one exception. A fork move is permitted if it is the only way to block an opponent from completing a row of five.

(2) A move that forms a row of more than five. In other words, the winning move must be *exactly* five.

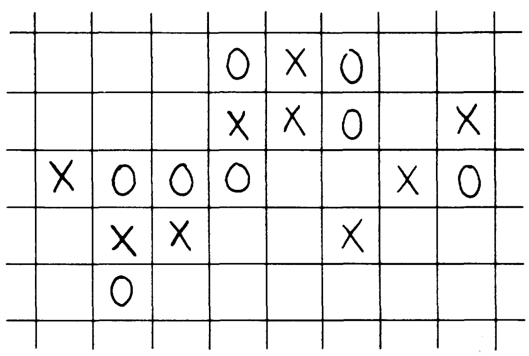
In master play, both rules are usually applied only to the first player. Under these rules, the game is commonly called "renju" in Japan.

It has been conjectured that if there is a winning strategy for the first player in unrestricted go-moku, on a large enough board, there will be a winning strategy on a sufficiently large board even if the prohibitions are observed, but this is far from established. Even if a winning strategy is found for unrestricted go-moku, difficult questions will remain. What is the *smallest* board on which the first player can win? What is the *shortest* way to win? The two questions may or may not be answered by the same line of play.

It is not possible that the second player has a winning strategy in unrestricted go-moku or similar games in any dimension. The bare bones of the simple *reductio ad absurdum* proof, first formulated by John Nash for the game of hex, are as follows. Assume that a second-player winning strategy exists. If it does, the first player can make an irrelevant, random first play—a play that can only be an asset—and then, since he is now in effect the second player, win by appropriating the second player's strategy. Because this contradicts the assumption, it follows that no second-player winning strategy exists. The first player can either win or at least force a draw if the game allows draws.

Go-moku is a stimulating game. To catch its special flavor the reader is urged to study a position from Silverman's book [*see Figure* 54] and determine how *O* can play and win in five moves. Note that *X* has an open-end diagonal of three, which he threatens to lengthen to an open-end row of four.

Figure 54



Go-moku problem: 0 to play and win

When ticktacktoe is extended to three dimensions, the first player wins easily on an order-3 cube by first taking the center cell. As Silverman points out, if the first player fails to open with the center cell, the second player can win by taking it; if the center is permanently prohibited to both players, the first player has an easy win. Three-dimensional toetacktick (the first row of three loses) is also a win for the first player. He plays the same strategy used for forcing a draw in planar toetacktick: He first takes the center and then always plays symmetrically opposite his opponent. Since drawn positions are impossible on the order-3 cube, this technique forces the second player eventually to form a row of three. Daniel I. A. Cohen, in a paper listed in the bibliography, proves that, as in the case of planar toetacktick, this is a unique winning strategy. The first player loses if he does not open by taking the central cell, and also loses if, after making this first move, he does not follow antipodal play.

Draw games *are* possible on the order-4 cube, but whether the first player can force a win is not, as far as I know, positively established. (There cannot, of course, be a second-player win because of Nash's proof.) As with go-moku, the first player has a strong advantage and a winning strategy is believed to exist. Many computer programs for this game have been written, but the complexity of play is so enormous that I do not think a first-player win has yet been rigorously demonstrated. About a dozen readers have sent me what they consider winning strategies, but detailed formal proofs are still unverified. Most of the strategies involve first taking four of the eight central cells and then proceeding to a forced win. Virtually nothing is known about three-dimensional games where counters are allowed to move from cell to cell.

Another unexplored type of 3-space game is one in which two players alternately draw from a limited supply of unit cubes of two or more colors to build a larger cube with some winning objective in view, for example, using cubes of n colors and trying to get a row, on an order-n cube, in which all n colors appear. For such games gravity imposes restraints, since cubes cannot be suspended in midair.

Because drawn games of standard ticktacktoe are possible in 2-space on an order-3 board, and in 3-space on an order-4 board, it was once conjectured that in a space of n dimensions the smallest board allowing a draw was one with $n+1$ cells on a side. It turned out, however, that although in n-space a board of order $n+1$ or higher always allows a draw, it is sometimes possible for an n-space board of fewer than $n+1$ cells on a side to allow a draw. This was first established about 1960 by Alfred W. Hales, when he constructed a draw on the order-4 hypercube, or fourth-dimension cube.

Several readers have sent informal but probably valid proofs that the first player can always win on the order-4 hypercube. Whether or not he can force a win on the order-5 hypercube is yet another of the many unanswered questions about extensions and variants of what most people, like Tom Sawyer, regard as a "simple" game.

ANSWERS

The second game in Figure 52 is not possible. Zero must have played first and last, but *X* had a win before the final move, so the last move would not have been made. In the third game, *X* could have completed a win if his first two moves had been on either side, therefore the first two moves must have been diagonally opposite, and his final move in the top right corner.

These two problems are so easily solved that I will add here a difficult one that involves what chess players call retrograde analysis. Figure 55 shows the pattern after two perfect players have agreed to a draw. Your task is to determine the first and last moves. If you can't solve it, you will find the solution in the *Journal of Recreational Mathematics,* Vol. 11, No. 1, 1978, page 70. The problem had been earlier posed in the same journal by Les Marvin.

Figure 55

X	O	O
		X
	X	O

What were the first and last moves?

In Silverman's first problem, *X* can always win, regardless of whether he plays first or second. Assume that the cells are numbered (left to right, top to bottom) from 1 to 9. Here is Silverman's proof:

If *X* begins, he takes 1. *O* must take 5, otherwise *X* can get three of his marks in a row by standard ticktacktoe strategy. *X*2 forces *O*3, then *X*4 forces *O*7, which completes three *O*'s in a line, giving *X* the win.

If *O* starts the game, he has a choice of corner, side or center opening. If he opens at the center (5), *X* responds with 1. If the move is *O*2, *X*7 forces *O*4, then *X*9 forces *O*8, which loses. If *O*'s second move is 3, *X*4 forces *O*7, which also loses. If *O*'s second move is 6, *X*7 forces *O* to lose at 4. If *O*'s second move is 9, *X*2 forces *O*3, then *X*4 forces *O* to lose at 7. All other lines of play are symmetrically equivalent.

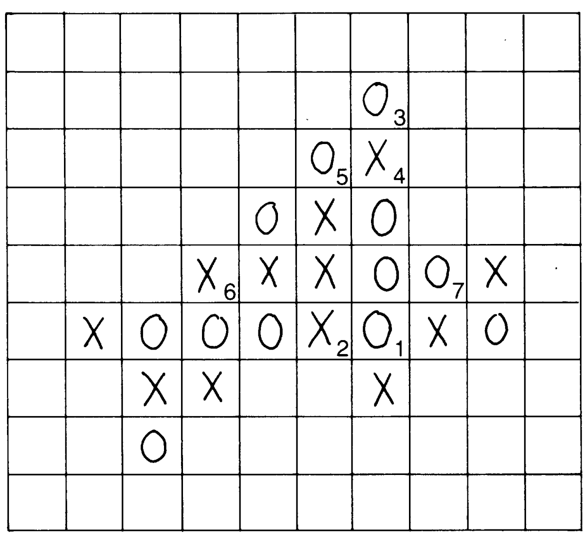
If *O* opens at the side, say at 4, *X*5 will win. As before, there are four basically different continuing lines of play: (1) *O*1, *X*3, *O*7 (loses), (2) *O*2, *X*3, *O*7, *X*9, *O*1 (loses), (3) *O*3, *X*9, *O*1, *X*8, *O*2 (loses), (4) *O*6, *X*3, *O*7, *X*9, *O*1 (loses).

A corner opening by *O,* say at 1, is met with *X*5, which leads again to four basically different continuations: (1) *O*2, *X*7, *O*3 (loses), (2) *O*3, *X*8, *O*2 (loses), (3) *O*6, *X*8, *O*2, *X*7, *O*3 (loses), (4) *O*9, *X*2, *O*8, *X*3, *O*7 (loses).

When this game is played on a four-by-four field (*X* winning if there are four of either mark in a row, *O* winning if the final position is drawn), the play is so enormously more complex, Silverman informs me, that it has not yet been fully analyzed.

O wins Silverman's go-moku problem by playing *O*1 [*see Figure* 56]. *X*2 is forced, *O*3 forces *X*4, *O*5 forces *X*6, then *O*7 creates an open-end diagonal row of four *O*'s, which *X* cannot block. If *X* plays at either end, *O* wins by playing at the opposite end. As Silverman points out in his book, *O* wins only by counterattacking. He loses quickly if he plays defensively by trying to block *X*'s open-end diagonal row of three.

Figure 56



Solution to the go-moku problem

Note that when *X* plays on the cell marked 2 it creates a fork. This is permitted, however, because the move is forced. It is the only way to prevent *O* from winning on the next move.

ADDENDUM

John Selfridge reports that a solution has been found for his "4 × infinity" ticktacktoe. This is played on a strip that is four cells high and infinitely wide, the winner being the first to get four of his marks in an orthogonal or diagonal row. Carlyle Lustenberger, in his master's thesis in computer science at Pennsylvania State University, developed a computer program with a winning strategy for the first player on a four-by-30 board. The actual lower bound for the width is a few cells shorter, but I have not obtained the details.

The three-by-infinity board is a trivial win for the first player on his third move; indeed, the same win can be achieved if only one cell is added to the side or corner cell of the traditional order-3 ticktacktoe field. The five-by-infinity board remains unsolved. If a win for the first player could be found on this board, it would, of course, solve the go-moku game when it is played on an arbitrarily large square, with no restrictive rules.

Oren Patashnik, of Bell Laboratories, was the first to write a computer program that establishes a first-player win in 4×4×4 ticktacktoe. I had the honor of announcing the verification of his 1977 program in my *Scientific American* column of January 1979. It required 1,500 hours of computing time, and has been likened to the computer proof of the four-color map theorem in its length and complexity. I will say no more about it here because Patashnik has so thoroughly and amusingly reported on it in his paper listed in the bibliography. The program's set of 2,929 strategic moves for winning is probably far from minimal, but I know of no program that has reduced them.

In 1973 the Netherlands issued a 30+10 cents stamp depicting a drawn pattern in a ticktacktoe game.

Shein Wang, a computer scientist at the University of Guelph, Guelph, Ontario, Canada, has been publishing a monthly *Gomoku Newsletter* since 1979, and the university has, since 1975, been sponsoring a North American computer go-moku tournament. The programs have been steadily improving.

A popular variation of go-moku is on sale in the United States under the trade name *Pente*. Invented by Gary Gabel, it combines go-moku with elements of go. (See *Newsweek,* May 10, 1982, page 78.)

Several readers wrote to emphasize that Nash's proof applies only to unrestricted go-moku. The proof rests on the irrelevance of an extra stone, but in restricted go-moku the rules permit situations in which an extra stone can damage the player who owns it.

Henry Pollak and Claude Shannon apparently were the first to prove that the second player can force a draw in unrestricted n-in-a-row ticktacktoe on a large enough board when $n=9$ or more. Their 1955 proof has not been published. It is given by T. G. L. Zetters in his answer to a problem, *American Mathematical Monthly*, Vol. 87, August–September 1980, pages 575–576. Zetters goes on to show how the proof can be extended to $n=8$ or more. So far as I know, the question is still open for $n=5$, 6 and 7.

W. F. Lunnon, writing in 1971 from University College, in Cardiff, gave a simple pairing strategy of unknown origin that guarantees a draw for the second player in 5×5 ticktacktoe. Number the cells as shown in Figure 57. Whenever the first player occupies a numbered cell, the second player takes the other cell of the same number. Since every line of five has a pair of like-numbered cells, the first player cannot occupy all five. If the first player takes the unlabeled center, the second player may take any cell, and if the cell he is required to take by the pairing strategy is occupied, he may play anywhere.

Lunnon also reported that he and Neil Sloane, of Bell Labs, had together found a remarkable second-player drawing strategy, based on cell pairing, for the 6×6 board. Not only does it

Figure 57

2	10	5	5	1
6	9	12	8	9
6	11		11	4
7	10	12	7	4
1	3	3	8	2

W. F. Lunnon's pairing strategy

block wins on any row, column or main diagonal, it also blocks a win on any broken diagonal! The cells are numbered as shown in Figure 58. As before, the strategy is to take the cell with the same number as the cell just taken.

Figure 58

1	13	2	13	3	12
6	14	5	14	4	12
7	8	15	9	10	15
16	3	11	1	16	2
17	4	11	6	17	5
7	8	18	9	10	18

Lunnon-Sloane second-player drawing strategy

There is more. The Lunnon-Sloane numbering leads to an elegant proof that 9-in-a-row unrestricted go-moku is a draw. Cover the infinite board with copies of the 6×6 matrix. The second player can force a draw by always taking the nearest cell with the same number as that of the previous play. It is easy to see that the first player can obtain no line longer than 8.

For $n \times n$ boards, n equal to 6 or higher, it is trivially easy to put a unique pair of numbers in each row of n cells and so provide a drawing strategy for the second player. For n equal to 3 or 4, no such labeling is possible, and the draw has to be established in uglier ways.

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