Exam 01 AY2023-24 Semester 1

Discrete Mathematics

2023-09-04

Part 1. Multiple Choice and Short Answer Questions

Problem 1. The first few Fibonacci numbers are $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$. Recall that Zeckendorf's theorem states that every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This sum is called the Zeckendorf representation of the number.

Let n be a positive integer and let $p_1 + p_2 + \cdots + p_k = n$ be its Zeckendorf representation, where $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$.

Let $m = n - p_1$ and let $q_1 + q_2 + \cdots + q_\ell = m$ be its Zeckendorf representation, where $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_\ell$. Assuming $k > 1$, it true that $q_1 > 2 \cdot p_1$?

☐ Yes ☐ No

Problem 2. How many sequences of length 5exist consisting only of numbers $0, 1, 2$such that each number occurs at least once? (Hint: use the principle of inclusion-exclusion.)

Problem 3. In this exercise, we will prove that **everyone is pretty much bald**. It goes by induction: we will prove that for all n, if you have n hairs on your head then you are pretty much bald. The base case is easy: if you have 1 hair on your head, then certainly you're pretty much bald.

Now suppose inductively that if you have n hairs on your head, then you're pretty much bald. We need to show that the same is true for someone with $n + 1$ hairs on their head. But certainly if someone has only 1 hair more on their head than someone else who is pretty much bald, then that first person is also pretty much bald. This completes the induction!

What can you say about this proof? It is possible that multiple options below strike you as being accurate, but please choose only one — the most appropriate according to you.

☐ It is an accurate proof of a true statement.

☐ The notion of *pretty much bald* has not been quantified, hence this does not count as a valid proof.

☐ The induction works only for the base case and breaks down when we move from 1 to 2.

☐ The base case is wrong.

Problem 4. Consider the two problems below.

(4.a) Is it possible to write a real number into each square of a 5×5 grid so that the sum of the numbers in the entire grid is negative, but the sum of the numbers in any 2×2 square (formed by 4 neighboring boxes) is positive?

☐ Yes ☐ No

(4.b) What about a 6×6 grid?

☐ Yes ☐ No

Problem 5. A heap consists of n stones. We split the heap into two smaller heaps, neither of which are empty. Denote p_1 the product of the number of stones in each of these two heaps. Now take any of the two small heaps, and do likewise. Let p_2 be the product of the number of stones in each of the two smaller heaps just obtained. Continue this procedure until each heap consists of one stone only. This will clearly take $n - 1$ steps. What is the value of the sum $p_1 + p_2 + \cdots + p_{n-1}$?

- ☐ $n(n+1)/2$ points
- ☐ n^2 points
- ☐ $n(n-1)/2$ points
- ☐ It depends on how the heaps were split.

Problem 6. What is the value of the sum $\sum_{k=2}^{n} k(k-1)\begin{pmatrix} n \\ k \end{pmatrix}$?

☐ $2^{n-2} \cdot n \cdot (n-1)$ ☐ $2^{n-1} \cdot n \cdot (n-1)$ ☐ $2^{n-2} \cdot (n-1) \cdot (n-2)$ ☐ 2^{n+1}

Problem 7. Consider the set S of all polynomials of finite degree with rational coefficients. Is there a bijection between S and $\mathbb{N}$, the set of natural numbers?

☐ Yes ☐ No

Problem 8. Are the two formulas below logically equivalent?

$$(A \wedge \bar{B} \wedge \bar{C}) \vee (\bar{A} \wedge B \wedge C) \vee (\bar{A} \wedge B \wedge \bar{C}) \vee (\bar{A} \wedge \bar{B} \wedge C) \vee (\bar{A} \wedge \bar{B} \wedge \bar{C})$$

and

$$\text{NOT}((A \wedge B) \vee (A \wedge C))$$

☐ Yes ☐ No

Problem 9. In the hexagon grid coloring game, we have a hexagonal grid whose top row has n cells, the next row and $n-1$ cells and so on; and the last row has just one cell. The top row is colored with one of three colors: red, blue, or green. We color a cell in any row subsequent based on the colors of its two neighboring cells above using the following rule:

- If the two neighboring cells on the previous row are colored with different colors, color the cell with the remaining color (`red` + `green` = `blue`; `red` + `blue` = `green`; `blue` + `green` = `red`).
- If the two neighboring cells on the previous row are colored with the same color, color the cell with the common color (`red` + `red` = `red`; `blue` + `blue` = `blue`; `green` + `green` = `green`).

Suppose the top row of a grid with $n = 28$ has the following colors:

`RBGRRBGRBRGBGBRBGBGRGBRBBGRG`

where `R`, `B` and `G` denote red, blue, and green respectively.

What is the color of the bottom cell?

☐ Red ☐ Blue ☐ Green

Part 2. Subjective Questions

Problem 10. Two people sit facing each other, call them Rohit and Babar; these are the players. A third person secretly writes two *consecutive* natural numbers on two slips of paper, and tapes each piece on the two players' foreheads (one on each). The third person then leaves the room (or sits quietly); his role in the game is finished.

Rohit can see the number taped to Babar's forehead, and likewise Babar can see the number taped to Rohit's forehead. So they both know the number that's not their own. They also both know that the two numbers are consecutive. They do *not* know their own number.

One player, say Rohit, begins the game by asking Babar:

Do you know your number?

If Babar knows (i.e, is able to infer) his number, he says YES and the game ends. If not, Rohit's turn ends and Babar gets his chance to ask Rohit:

Do you know your number?

As before, if Rohit knows (i.e, is able to infer) his own number, then he says YES and the game ends.

Otherwise, it becomes Rohit's turn again, and he repeats his original question to Babar. This back and forth questioning continues until someone finally says YES, if ever.

Let n denote the lower of these two numbers. Prove by induction on n that the game will end in no more than $2n$ turns.

Hint: Think about the base case and work out the game play for some small values of n.

Problem 11. A group of n unemployed mathematicians aligned themselves and formed an international network of math-thieves. After a particularly successful heist, the group found themselves in possession of ten lakh rupees. A meeting was called to distribute the money to the members. Each member has a unique rank in the organization, from 1st ranked (the leader) all the way down to n^{th} ranked (the last-in-command).

As it turns out, a very precise code is in place that governs how surplus income is to be distributed. To begin with, the 1st ranked member decides on a potential distribution of the wealth. Each member must be assigned a whole rupee amount (no paise), with 0 rupees of course being allowed. This potential distribution is then put to a secret vote, wherein each member, including the leader, gets to cast exactly one ballot: `Yes` or `No`. The members cannot communicate or strategize amongst themselves; it is every ex-mathematician for themselves.

If the vote passes or is a tie, then the money is distributed according to the proposed distribution. The catch is this: if the vote fails, then the 1st ranked member is ousted from the organization forever. Every other member is promoted by exactly one rank to fill the power vacuum, and the new 1st ranked member (who used to be 2nd ranked) repeats the process by indicating a new potential distribution and putting it to a vote. This continues until one of the distributions is passed, at which point the members take whatever money was allotted to them by that distribution.

Each member is very invested in this international network, and would rather get no share of the money at all than be ousted from the organization. Each member would also prefer not to oust too many people, if possible, so if all else is equal (i.e. if they would get the same payoff either way), then a member will vote `Yes` rather than `No` on a given distribution. Of course, if they figure that they can get even a single extra rupee by voting `No` on the current plan, they will do it. That's the way the world works, at least among secret math thieves.

How much cash can the leader pocket?

End of the exam