Exam 01 AY2023-24 Semester 1

Discrete Mathematics

2023-09-04

Part 1. Multiple Choice and Short Answer Questions

Problem 1. The first few Fibonacci numbers are $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$. Recall that Zeckendorf's theorem states that every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This sum is called the Zeckendorf representation of the number.

Let n be a positive integer and let $p_1 + p_2 + \cdots + p_k = n$ be its Zeckendorf representation, where $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k$.

Let $m = n - p_1$ and let $q_1 + q_2 + \cdots + q_\ell = m$ be its Zeckendorf representation, where $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_\ell$. Assuming $k > 1$, it true that $q_1 > 2 \cdot p_1$?

■ **Yes** ☐ No

If $m = n - p_1$, then $m = p_2 + \cdots + p_k$ is the Zeckendorf representation of m. Therefore, $q_1 = p_2$. However, if $p_1 = f_i$, the i^{th} Fibonacci number, and $p_2 = f_j$, the j^{th} Fibonacci number, then we know that $j > i + 1$, and:

$$q_1 = p_2 = f_j = f_{j-1} + f_{j-2} \geqslant f_{j-1} + f_i > f_i + f_i = 2 \cdot f_i = 2 \cdot p_1,$$

as required.

Problem 2. How many sequences of length 5exist consisting only of numbers $0, 1, 2$such that each number occurs at least once? (Hint: use the principle of inclusion-exclusion.)

The total number of such sequences is 243; the number of sequences that miss 0 is 32; the number of sequences that miss 1 is 32; the number of sequences that miss 2 is 32. So we have a candidate answer in $243 - 32 * 3 = 147$, but notice that the strings `00000`, 11111, and 22222 have been subtracted twice each, so the correct count is 150.

Problem 3. In this exercise, we will prove that **everyone is pretty much bald**. It goes by induction: we will prove that for all n, if you have n hairs on your head then you are pretty much bald. The base case is easy: if you have 1 hair on your head, then certainly you're pretty much bald.

Now suppose inductively that if you have n hairs on your head, then you're pretty much bald. We need to show that the same is true for someone with $n + 1$ hairs on their head. But certainly if someone has only 1 hair more on their head than someone else who is pretty much bald, then that first person is also pretty much bald. This completes the induction!

What can you say about this proof? It is possible that multiple options below strike you as being accurate, but please choose only one — the most appropriate according to you.

☐ It is an accurate proof of a true statement.

- ■ **The notion of *pretty much bald* has not been quantified, hence this does not count as a valid proof.**
- ☐ The induction works only for the base case and breaks down when we move from 1 to 2.
- ☐ The base case is wrong.

The notion of *pretty much bald* has not been quantified. If you say that you are *pretty much bald* if you have, say, less than h hairs on your head, then this proof will break down at a very specific point for any choice of h. This is similar to the "proof" that every number is *much smaller* than one billion. Adapted from here.

Problem 4. Consider the two problems below.

(4.a) Is it possible to write a real number into each square of a 5×5 grid so that the sum of the numbers in the entire grid is negative, but the sum of the numbers in any 2×2 square (formed by 4 neighboring boxes) is positive?

■ **Yes** ☐ No

For 5×5 grids, highlight alternate rows and columns, and fill the intersections of the highlighted rows and columns with 4, and set all other cells to -1. This has the desired property.

(4.b) What about a 6×6 grid?

☐ Yes ■ **No**

For 6×6 grids, however, the answer is no. Indeed, if B is a 6×6 grid, then B can be partitioned into nine squares of size 2×2 each, in an obvious way. Then the sum of the elements of B must equal that of the sum of elements of these 2×2 squares.

Problem 5. A heap consists of n stones. We split the heap into two smaller heaps, neither of which are empty. Denote p_1 the product of the number of stones in each of these two heaps. Now take any of the two small heaps, and do likewise. Let p_2 be the product of the number of stones in each of the two smaller heaps just obtained. Continue this procedure until each heap consists of one stone only. This will clearly take $n-1$ steps. What is the value of the sum $p_1 + p_2 + \cdots + p_{n-1}$?

- ☐ $n(n+1)/2$ points
- ☐ n^2 points
- ■ $n(n-1)/2$ **points**
- ☐ It depends on how the heaps were split.

This sum is always the same, namely, it is $\binom{n}{2}$ if $n > 1$. We prove this by strong induction on n. The initial case is trivial. Assume we know the statement for all positive integers less than n, and prove it for n. Let us split our heap of n stones into two small heaps, one of size k, and one of size $n-k$. Then $p_1 = k(n-k)$. Then, by our induction hypothesis, the contribution of the first heap to the sum $p_1 + p_2 + \cdots + p_{n-1}$ is $\binom{k}{2}$, and that of the second heap is $\binom{n-k}{2}$.

As

$$k(n-k) + \binom{k}{2} + \binom{n-k}{2} = \binom{n}{2},$$

our claim is proved.

This explanation is borrowed from *A Walk through Combinatorics*, second edition, by Miklós Bóna. A game based on this features here.

Problem 6. What is the value of the sum $\sum_{k=2}^{n} k(k-1)\binom{n}{k}$?

■ $2^{n-2} \cdot n \cdot (n-1)$ □ $2^{n-1} \cdot n \cdot (n-1)$ □ $2^{n-2} \cdot (n-1) \cdot (n-2)$ □ 2^{n+1}

Both expressions count the number of ways in which a committee can be chosen out of n people that has a president and a vice-president, with both roles being fulfilled by distinct individuals.

Problem 7. Consider the set S of all polynomials of finite degree with rational coefficients. Is there a bijection between S and $\mathbb{N}$, the set of natural numbers?

■ **Yes** □ No

Note that all polynomials of finite degree with rational coefficients are in one-to-one correspondence with the set of all finite sequences of natural numbers. Consider the mapping:
$(c_1, \ldots, c_d) \longrightarrow p_1^{c_1} \cdot p_2^{c_2} \cdots p_d^{c_d}$,
where p_i is the i^{th} prime. This is clearly an injection from $\cup_{i=1}^{\infty} \mathbb{N}^d \rightarrow \mathbb{N}$. The identitfy function is a trival injection from $\mathbb{N} \rightarrow \cup_{i=1}^{\infty} \mathbb{N}^d$. Therefore, there exists a bijection between finite sequences of natural numbers and natural numbers.

Problem 8. Are the two formulas below logically equivalent?

$$(A \wedge \bar{B} \wedge \bar{C}) \vee (\bar{A} \wedge B \wedge C) \vee (\bar{A} \wedge B \wedge \bar{C}) \vee (\bar{A} \wedge \bar{B} \wedge C) \vee (\bar{A} \wedge \bar{B} \wedge \bar{C})$$

and

$$\text{NOT}((A \wedge B) \vee (A \wedge C))$$

■ **Yes** □ No

These propositions are equivalent and this can be established by either working through simplifications based on De Morgan's laws or using truth tables. A detailed worked out argument is given here just before Theorem 3.4.2 (p.61).

Problem 9. In the hexagon grid coloring game, we have a hexagonal grid whose top row has n cells, the next row and $n-1$ cells and so on; and the last row has just one cell. The top row is colored with one of three colors: red, blue, or green. We color a cell in any row subsequent based on the colors of its two neighboring cells above using the following rule:

- If the two neighboring cells on the previous row are colored with different colors, color the cell with the remaining color (`red` + `green` = `blue`; `red` + `blue` = `green`; `blue` + `green` = `red`).
- If the two neighboring cells on the previous row are colored with the same color, color the cell with the common color (`red` + `red` = `red`; `blue` + `blue` = `blue`; `green` + `green` = `green`).

Suppose the top row of a grid with $n = 28$ has the following colors:

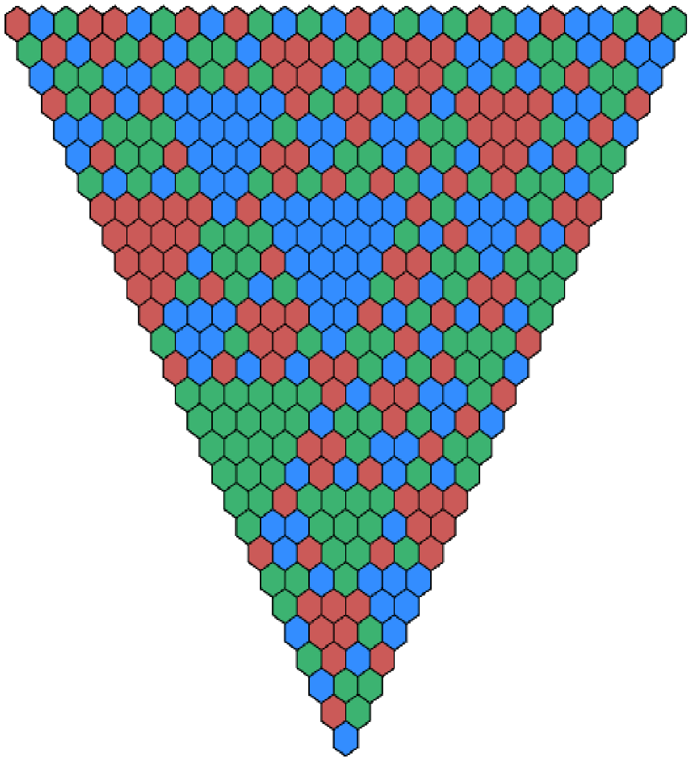
`RBGRRBGRBRGBGBRBGBGRGBRBBGRG`

where `R`, `B` and `G` denote red, blue, and green respectively.

What is the color of the bottom cell?

□ Red ■ **Blue** □ Green

It can be shown that if the top row has $3^n + 1$ cells, then the answer is the rule applied to the first and the last cell. Since $28 = 3^3 + 1$, the answer is `R + G = B`. You can double-check that this indeed checks out by powering through the rule!



Part 2. Subjective Questions

Problem 10. Two people sit facing each other, call them Rohit and Babar; these are the players. A third person secretly writes two *consecutive* natural numbers on two slips of paper, and tapes each piece on the two players' foreheads (one on each). The third person then leaves the room (or sits quietly); his role in the game is finished.

Rohit can see the number taped to Babar's forehead, and likewise Babar can see the number taped to Rohit's forehead. So they both know the number that's not their own. They also both know that the two numbers are consecutive. They do *not* know their own number.

One player, say Rohit, begins the game by asking Babar:

> *Do you know your number?*

If Babar knows (i.e, is able to infer) his number, he says YES and the game ends. If not, Rohit's turn ends and Babar gets his chance to ask Rohit:

> *Do you know your number?*

As before, if Rohit knows (i.e, is able to infer) his own number, then he says YES and the game ends.

Otherwise, it becomes Rohit's turn again, and he repeats his original question to Babar. This back and forth questioning continues until someone finally says YES, if ever.

Let n denote the lower of these two numbers. Prove by induction on n that the game will end in no more than $2n$ turns.

Hint: Think about the base case and work out the game play for some small values of n.

Let the set of natural numbers be $\mathbb{N} = \{1, 2, 3, ...\}$ (a similar argument works if you believe $0 \in \mathbb{N}$).

When Rohit has 1 and Babar has 2, the claim is that the game ends in at most two turns.

- If Rohit has the first question, Babar actually answers YES, and we are immediately done.
- If Babar has the first question, Rohit answers NO since (he's uncertain between 1 and 2), but on the next turn, Babar answers YES, and we are done in two rounds.

The same argument — with names swapped — works if Rohit has 2 and Babar has 1.

When Rohit has 2 and Babar has 3, the claim is that the game ends in at most four turns.

Suppose Rohit asks the first question.

- Turn 1. Babar answers NO, since Babar is uncertain between $\{1, 3\}$.
- Turn 2. Rohit answers NO, since Rohit is uncertain between $\{2, 4\}$.
- Turn 3. Babar answers YES, because he can deduce that if he had a 1, then Rohit would not have said no.

So the game ends in three rounds. If Babar had the first question, there would be one additional turn as Rohit would answer NO on Turn 3.

The same argument — with names swapped — works if Rohit has 3 and Babar has 2.

In general, we propose the following induction hypothesis:

- If the numbers written are n for Rohit and $n+1$ for Babar, and Rohit asks the first question, the game ends in at most $2n-1$ rounds.
- If the numbers written are n for Rohit and $n+1$ for Babar, and Babar asks the first question, the game ends in at most $2n$ rounds.
- If the numbers written are $n+1$ for Rohit and n for Babar, and Babar asks the first question, the game ends in at most $2n-1$ rounds.
- If the numbers written are $n+1$ for Rohit and n for Babar, and Rohit asks the first question, the game ends in at most $2n$ rounds.

Notice that the last two statements are the same as the first two with the names swapped.

Assuming the induction hypothesis is true, we have to prove the following:

- If the numbers written are $n+1$ for Rohit and $n+2$ for Babar, and Rohit asks the first question, the game ends in at most $2(n+1)-1 = 2n+1$ rounds.
- If the numbers written are $n+1$ for Rohit and $n+2$ for Babar, and Babar asks the first question, the game ends in at most $2(n+1) = 2n+2$ rounds.
- If the numbers written are $n+2$ for Rohit and $n+1$ for Babar, and Babar asks the first question, the game ends in at most $2(n+1)-1 = 2n+1$ rounds.
- If the numbers written are $n+2$ for Rohit and $n+1$ for Babar, and Rohit asks the first question, the game ends in at most $2(n+1) = 2n+2$ rounds.

Suppose we are in the first scenario. Then the game starts as follows.

- Turn 1. Babar answers `NO`, since Babar is uncertain between $\{n, n+2\}$.
- Turn 2. Rohit answers `NO`, since Rohit is uncertain between $\{n+1, n+3\}$.

Suppose we complete $2n$ rounds and the game is not yet over. That means the last turn involved Babar asking a question and Rohit saying `NO`. By now, Babar knows that his number could not have been n, because we know that:

> If the numbers written are $n+1$ for Rohit and n for Babar, and Rohit asks the first question, the game ends in at most $2n$ rounds.

In the very next turn, Rohit asks a question, and Babar says `YES`.

The other scenarios can be argued similarly.

You can find another explanation here.

Problem 11. A group of n unemployed mathematicians aligned themselves and formed an international network of math-thieves. After a particularly successful heist, the group found themselves in possession of ten lakh rupees. A meeting was called to distribute the money to the members. Each member has a unique rank in the organization, from 1st ranked (the leader) all the way down to n^{th} ranked (the last-in-command).

As it turns out, a very precise code is in place that governs how surplus income is to be distributed. To begin with, the 1st ranked member decides on a potential distribution of the wealth. Each member must be assigned a whole rupee amount (no paise), with 0 rupees of course being allowed. This potential distribution is then put to a secret vote, wherein each member, including the leader, gets to cast exactly one ballot: `Yes` or `No`. The members cannot communicate or strategize amongst themselves; it is every ex-mathematician for themselves.

If the vote passes or is a tie, then the money is distributed according to the proposed distribution. The catch is this: if the vote fails, then the 1st ranked member is ousted from the organization forever. Every other member is promoted by exactly one rank to fill the power vacuum, and the new 1st ranked member (who used to be 2nd ranked) repeats the process by indicating a new potential distribution and putting it to a vote. This continues until one of the distributions is passed, at which point the members take whatever money was allotted to them by that distribution.

Each member is very invested in this international network, and would rather get no share of the money at all than be ousted from the organization. Each member would also prefer not to oust too many people, if possible, so if all else is equal (i.e. if they would get the same payoff either way), then a member will vote `Yes` rather than `No` on a given distribution. Of course, if they figure that they can get even a single extra rupee by voting `No` on the current plan, they will do it. That's the way the world works, at least among secret math thieves.

How much cash can the leader pocket?

We claim that the leader takes all, and prove this by induction. The leader's proposed distribution is that the leader keeps all the monies to herself.

If $n = 1$, then the leader certainly takes all.

If $n = 2$, the leader votes `Yes`, the second-ranked vote `No`, and the tie works in favor of the leader.

If $n = 3$, the leader votes `Yes`, the second-ranked vote `No`, and the third-ranked votes `Yes`, because she's getting nothing either way.

Let the induction hypothesis be that if there are k unemployed mathematicians, then the leader can propose to distribute the entire wealth to herself and will get to keep it.

Suppose $n = k + 1$, and the leader proposes to distribute the entire wealth to herself. Then the second-ranked member of the gang votes `No`, but all others realize that if the leader is ousted, then there will be k mathematicians and they will get nothing because of the Induction Hypothesis, so everyone else votes `Yes`, and the leader takes all.

Variations on this theme make the problem more challenging:

- How much cash can the leader pocket if tie votes result in oustings?
- How much cash can the leader pocket if members vote `No` rather than `Yes` if they get the same payoff either way?

Source (among others).