Lecture 6 - Max Cut via SDP

Sunday, November 20, 2022 20:12

SEMI-DEFINITE PROGRAMMING

Max / Min: $\sum_{i,j} c_{ij} (v_i \cdot v_j)$ Select $v_1, \dots, v_n \in \mathbb{R}^n$

s.t. $\forall k: \sum_{i,j} a_{i,j,k} (v_i \cdot v_j) = b_k$

$\forall i: v_i \in \mathbb{R}^n$

Generalizes LP? → Max/Min $\sum_i c_i (v_i \cdot v_i)$

s.t. $\forall k \sum_i a_i (v_i \cdot v_i) = b_k$

only $=$, what about $\leq$?

slack variables!

Thm: Can solve SDP in poly time (Kind of)

Main power:
can do "AND" of 2 vars ... KIND OF

Application: Max Cut. approximation!

Max Cut IQP

Max: $\sum_{v_i v_j \in E} \frac{1}{2} (1 - x_i \cdot x_j)$ (1)

st $\forall i \; x_i \in \{-1, 1\}$

Note: $\forall$ cut (A, B) setting $x_i = \begin{cases} 1 & \text{if } v_i \in A \\ -1 & \text{if } v_i \in B \end{cases}$

- gives IQP soln with value $\partial(A)$
- $\forall$ assignment $x_1 \dots x_n$ setting $A = \{v_i : x_i = 1\}$ gives cut [with same value]

But we can solve SDP not IQP!

↳ Relaxation:

Max $\frac{1}{2} \sum 1 - v_i v_j$

$\forall i \; v_i \cdot v_i = 1$ (2)

$v_i \in \mathbb{R}^n$

Relaxation?

Given soln $x_1 \dots x_n$ set $v_i = [x_i, 0, 0, 0 \dots 0]$

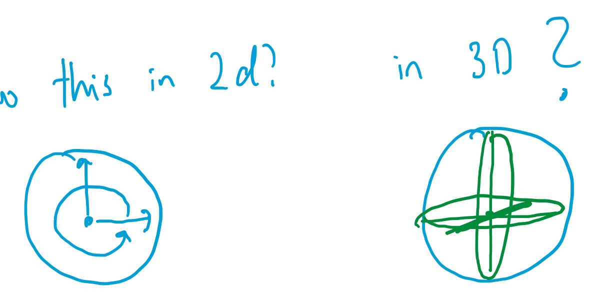
. feasible

Same value! ∎

Given soln $v_1 \dots v_n$, how to find soln to (1)? $x_1 \dots x_n \in \{-1, 1\}$

Algorithm: Pick a random vector $r \in \mathbb{R}^n$ st $|r| = 1$ ($r \cdot r$)

How to do this in 2d?   in 3D?



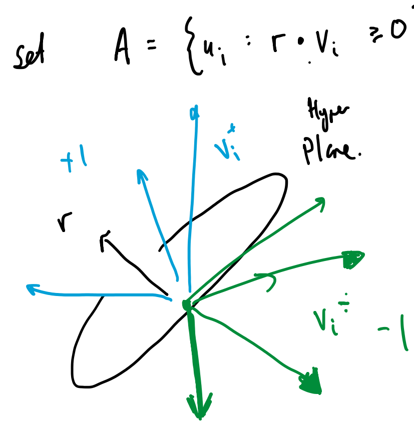
Trick: Let $\hat{r} = [r_1, r_2 \dots r_n]$ where $r_i \sim N(0,1)$

and $r = \hat{r} / |\hat{r}|$

(won't see why this is uni)

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

set $A = \{u_i : r \cdot v_i \geq 0\}$

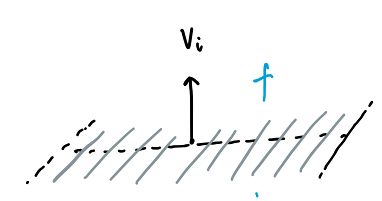


A defines a cut... but how good is it [in expectation?]

$$E[|\partial(A)|] = \sum_{u_i u_j \in E} E\left[\begin{smallmatrix} 1 & \text{if } u_i u_j \in \partial(A) \\ 0 & \text{otherwise} \end{smallmatrix}\right]$$

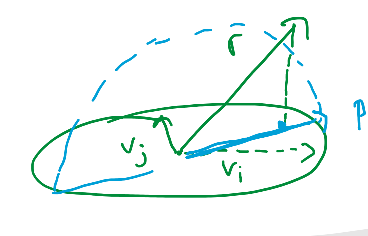
$$= \sum_{u_i u_j} Pr[u_i u_j \in \partial(A)]$$

$$= Pr[\text{sign}(r \cdot v_i) \neq \text{sign}(r \cdot v_j)]$$



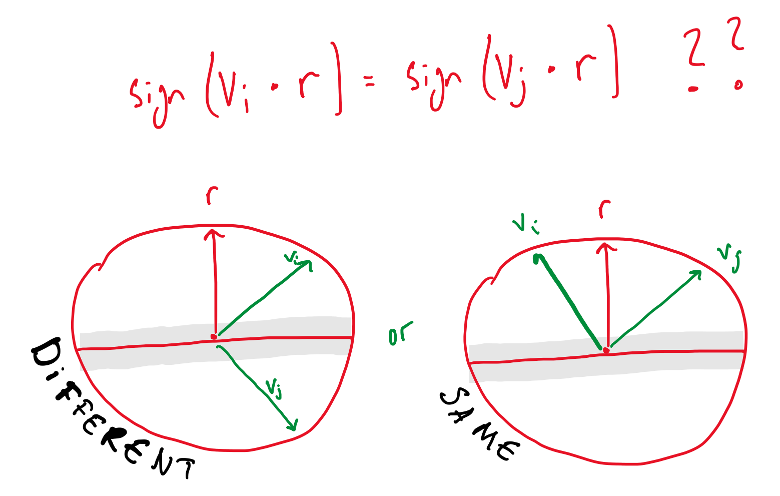
Same for v_j.

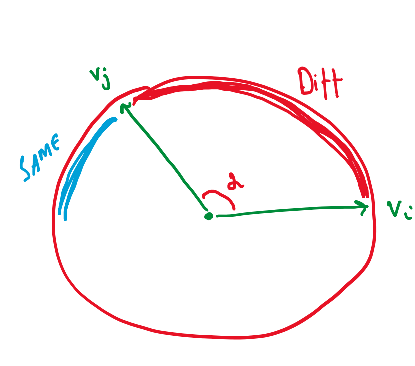
OBS: $\text{proj}(r, \text{Span}(v_i, v_j))$ is uniform over the $v_i - v_j$ circle.



$\text{sign}(v_i \cdot r) = \text{sign}(v_i \cdot \text{proj}(r, \text{span}(v_i, v_j)))$

$\text{sign}(v_i \cdot r) = \text{sign}(v_j \cdot r)$??





$v_i \cdot v_j = |v_i| \cdot |v_j| \cdot \cos(\alpha) = \cos(\alpha)$

$$P[u_i u_j \in \partial(A)] = \frac{\alpha}{\pi} = \frac{\arccos(v_i \cdot v_j)}{\pi}$$

$$E[\partial(A)] = \sum_{u_i u_j} \frac{\arccos(v_i v_j)}{\pi}$$

$$\geq \sum_{u_i u_j} 0.87 \cdot \frac{1}{2}(1 - v_i v_j)$$

$$= 0.87 \cdot OPT_{SDP} \geq 0.87 \, OPT$$